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***N*-species stochastic models with boundaries and quadratic algebras**

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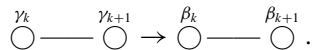
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Abstract. Stationary probability distributions for stochastic processes on linear chains with closed or open ends are obtained using the matrix product ansatz. The matrices are representations of some quadratic algebras. The algebras and the types of representations considered depend on the boundary conditions. In the language of quantum chains we obtain the ground state of *N*-state quantum chains with free boundary conditions or with non-diagonal boundary terms at one or both ends. In contrast to problems involving the Bethe ansatz, we do not have a general framework for arbitrary *N*, which when specialized, gives the known results for *N* = 2; in fact, the *N* = 2 and *N* > 2 cases appear to be very different.

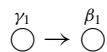
1. Introduction

The aim of this paper is to present in a systematic way the application of quadratic algebras to obtain the steady-state probability distribution of one-dimensional stochastic processes with boundaries. In this section we first present the problem and describe what is already known about the subject giving the relevant references and then proceed by giving the contents of the next sections. The list of possible physical applications of our results includes interface growth [1], boundary induced phase transitions [2–5], the dynamics of shocks [6] or traffic flow [7].

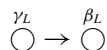
Consider a linear chain with *L* sites. On each site *k* we put a discrete stochastic variable β_k taking values from $\{0, 1, \dots, N - 1\}$. For each link *k* between sites *k* and *k* + 1 we give transition (intensity) rates $\Gamma_{\beta_k \beta_{k+1}}^{\gamma_k \gamma_{k+1}}$ giving the probability per unit time for the transition



These are the *bulk rates*. At the left end of the chain (site 1) we also consider transition rates



given by the *N* × *N* matrix $L_{\beta_1}^{\gamma_1}$. Similarly, at the right end of the chain (site *L*) we take processes



with rates $R_{\beta_L}^{\gamma_L}$. The matrices L and R give the *boundary rates*. We consider Markov processes in continuous time, which implies that the rate matrices are *intensity* matrices, which have the property [8]

$$\Gamma_{\beta_k \beta_{k+1}}^{\beta_k \beta_{k+1}} = - \sum_{\substack{\gamma_k, \gamma_{k+1}=0 \\ (\gamma_k, \gamma_{k+1}) \neq (\beta_k, \beta_{k+1})}}^{N-1} \Gamma_{\gamma_k \gamma_{k+1}}^{\beta_k \beta_{k+1}} \quad L_{\beta}^{\beta} = - \sum_{\substack{\gamma=0 \\ \gamma \neq \beta}}^{N-1} L_{\gamma}^{\beta} \quad R_{\beta}^{\beta} = - \sum_{\substack{\gamma=0 \\ \gamma \neq \beta}}^{N-1} R_{\gamma}^{\beta}. \quad (1.1)$$

Sometimes it will be convenient to write the intensity matrices (for example R) in an alternative form

$$R_{\beta}^{\gamma} = R_{\beta_{\gamma}} \quad (1.2)$$

indicating by the column (γ) the initial state and by the row (β) the final state. Equations (1.1) just state that the sum of the matrix elements on each column of an intensity matrix is zero. The time evolution of the probability distribution $P(\beta_1, \beta_2, \dots, \beta_L; t) =: P(\{\beta\}; t)$ is given by the master equation

$$\begin{aligned} \frac{\partial}{\partial t} P(\{\beta\}; t) &= \sum_{k=1}^{L-1} \sum_{\gamma_k, \gamma_{k+1}=0}^{N-1} \Gamma_{\beta_k \beta_{k+1}}^{\gamma_k \gamma_{k+1}} P(\beta_1, \beta_2, \dots, \gamma_k, \gamma_{k+1}, \dots, \beta_L; t) \\ &+ \sum_{\gamma_1=0}^{N-1} L_{\beta_1}^{\gamma_1} P(\gamma_1, \beta_2, \dots, \beta_L; t) + \sum_{\gamma_L=0}^{N-1} R_{\beta_L}^{\gamma_L} P(\beta_1, \dots, \beta_{L-1}, \gamma_L; t). \end{aligned} \quad (1.3)$$

Solving equation (1.3) is equivalent to finding the wavefunction of an imaginary time Schrödinger equation (see [9] and references therein for the notation) which is obtained as follows. We consider an orthonormal system of states

$$|\beta\rangle = |\beta_1, \dots, \beta_L\rangle \quad \langle\gamma| \cdot |\beta\rangle = \prod_{j=1}^L \delta_{\gamma_j \beta_j} \quad (1.4)$$

and a basis in the space of $N \times N$ matrices $E^{\alpha\beta}$:

$$(E^{\alpha\beta})_{\gamma, \delta} = \delta_{\alpha, \gamma} \delta_{\beta, \delta}. \quad (1.5)$$

By $E_k^{\alpha\beta}$ we denote the matrix $E^{\alpha\beta}$ acting on the k th site. The probability distribution $P(\{\beta\}; t)$ is mapped into a ket state

$$|P\rangle = \sum_{\{\beta\}} P(\{\beta\}; t) |\beta\rangle = \begin{pmatrix} P(0, 0, \dots, 0; t) \\ P(0, 0, \dots, 1; t) \\ \vdots \\ P(N-1, N-1, \dots, N-1; t) \end{pmatrix} \quad (1.6)$$

and the master equation (1.3) implies the imaginary time Schrödinger equation

$$\frac{\partial}{\partial t} |P\rangle = -H |P\rangle \quad (1.7)$$

where

$$H = \sum_{k=1}^{L-1} H_{k, k+1} + H_1 + H_L \quad (1.8)$$

$$H_{k, k+1} = -\Gamma_{\gamma\delta}^{\alpha\beta} E_k^{\gamma\alpha} E_{k+1}^{\delta\beta} \quad (1.9)$$

$$H_1 = -L_{\beta}^{\alpha} E_1^{\beta\alpha} \quad \text{and} \quad H_L = -R_{\beta}^{\alpha} E_L^{\beta\alpha}. \quad (1.10)$$

Above, and subsequently, the summation is invoked on *any* pair of repeated indices (raised, lowered or mixed).

In this paper we are interested in finding the stationary probability distribution $P_s(\{\beta\})$ of the master equation (1.3), i.e. the ground-state wavefunction $|P_s\rangle\rangle$ of the Hamiltonian (1.8):

$$H|P_s\rangle\rangle = 0. \tag{1.11}$$

Since H is an intensity matrix with positive rates the ground-state energy is zero [10]. Unless stated otherwise we consider unnormalized probability distributions. The interest in knowing the stationary distributions of stochastic processes with boundaries is illustrated by the extensive list of references on the subject which can be found, for example, in [5]. If the bulk rates give a Hamiltonian with a certain algebra or superalgebra as symmetry (see [11] for many examples) the boundary terms might break this symmetry.

Recently, Krebs and Sandow [12] have proven the following remarkable theorem. With Γ , L , R matrices as above, take N matrices D_α ($\alpha = 0, 1, \dots, N - 1$) and N matrices X_α ($\alpha = 0, 1, \dots, N - 1$) acting in an *auxiliary* vector space which satisfy the quadratic algebra relations†

$$\Gamma_{\gamma\delta}^{\alpha\beta} D_\alpha D_\beta = D_\gamma X_\delta - X_\gamma D_\delta \quad (\gamma, \delta = 0, 1, \dots, N - 1). \tag{1.12}$$

Consider now a ket state $|0\rangle$ and a bra state $\langle 0|$ in the auxiliary space (we use the same notation suggesting the ‘vacuum’ of quantum mechanics for reasons which will be apparent later). If these states are chosen such that the following conditions are fulfilled

$$\langle 0|(X_\alpha - L_\alpha^\beta D_\beta) = 0 \quad \text{and} \quad (X_\alpha + R_\alpha^\beta D_\beta)|0\rangle = 0 \tag{1.13}$$

then

$$\begin{aligned} P_s(\{\beta\}) &= \langle 0| \prod_{k=1}^L D_{\beta_k} |0\rangle \\ &= \langle 0| \prod_{k=1}^L \left(\sum_{\mu_k=0}^{N-1} D_{\mu_k} \delta_{\beta_k, \mu_k} \right) |0\rangle \end{aligned} \tag{1.14}$$

is a stationary solution of the master equation (1.3). Alternatively, let us denote by u_{μ_k} ($\mu = 0, 1, \dots, N - 1 : k = 1, \dots, L$) the basis vectors in the vector space associated with the k th site ($(u_{\mu_k})_v$ for any k) on which the matrices $E_k^{\beta\alpha}$ act (see equations (1.9) and (1.10)). The ground-state wavefunction of the Hamiltonian (1.8) has the expression

$$|P_s\rangle\rangle = \langle 0| \prod_{k=1}^L \left(\sum_{\mu_k=0}^{N-1} D_{\mu_k} u_{\mu_k} \right) |0\rangle. \tag{1.15}$$

In fact, the Krebs–Sandow theorem is even stronger; the matrices $\Gamma_{\gamma\delta}^{\alpha\beta}$, R_β^α and L_β^α in (1.8) do not have to be intensity matrices. If the Hamiltonian has an eigenvalue zero, then the wavefunction is given by (1.15). Moreover, Krebs and Sandow have shown that the algebra (1.12) exists, giving a representation for the matrices D_α and X_α . This representation fulfils the second part of equation (1.13) but not the first.

In order to compute the ground-state wavefunction one needs the ‘vacuum’ expectation values of monomials of the form

$$\langle 0| D_{\mu_1}^{r_1} D_{\mu_2}^{r_2} \dots D_{\mu_s}^{r_s} |0\rangle \tag{1.16}$$

which are obtained from equations (1.12) and (1.13). In order to find the wavefunction of the two-site problem one considers ‘vacuum’ expectation values of monomials of degree

† Observe that the quadratic algebra (1.12) is unusual since the X_α ’s appear linearly. (See [13] and [14] for a discussion of quadratic algebras in the mathematical literature.)

two which are obtained from a system of linear equations. The solution of this system is not necessarily unique since the ground state might be degenerate. On completing this exercise the $L = 3$ ‘vacuum’ expectation values (1.16) are considered, and so on. This entire process, however, is more complicated than diagonalizing the Hamiltonian by brute force, and there should be a better way of solving the problem. This is indeed possible if the intensity matrices appearing in the Hamiltonian satisfy certain conditions. At this point, we shall mention only what is known in the $N = 2$ case. In [15] Hinrichsen *et al* guessed a four-dimensional representation for the matrices D_0 , D_1 , X_0 and X_1 for a specific choice of intensity matrices and derived certain concentration profiles. In this specific case the problem can also be solved directly since the Hamiltonian can be written in terms of free fermions and it was this solution which inspired the guess. This reference is remarkable since it is the first place where the relations (1.12)–(1.16) appear.

We shall now describe the cases where the representation theory is understood. Consider the case where the only non-zero bulk rates are Γ_{01}^{10} and Γ_{10}^{01} but leave the surface (boundary) rates L_1^0 , L_0^1 and R_1^0 , R_0^1 arbitrary. This is the asymmetric exclusion model with open boundaries. One can choose $X_1 = -X_0 = 1$ (the c -number 1) in equations (1.12) and (1.13) and, after a linear transformation $D \rightarrow Y$ which includes constants related to the boundary rates, bring the relations to the form

$$[Y_0, Y_1] = \sum_{\gamma, \delta \in \{0,1\}} c^{\gamma\delta} Y_\gamma Y_\delta + \sum_{\gamma \in \{0,1\}} c^\gamma Y_\gamma + c_{01} \quad (1.17)$$

where $[A, B] = AB - BA$ and $c^{\gamma\delta} = c^{\delta\gamma}$. The parameters $c^{\gamma\delta}$, c^γ and c_{01} depend on the bulk and surface rates and we also obtain

$$\langle 0|Y_1 = 0 \quad Y_0|0\rangle = 0 \quad \langle 0|0\rangle \neq 0. \quad (1.18)$$

The quadratic algebra with two generators (1.17) is well understood [14] and the Fock representations defined by equation (1.18) are known [4, 9]. It was in the work of Derrida *et al* [4] that Fock representations of quadratic algebras were used for the first time to find ground-state wavefunctions and to compute correlation functions. As one can notice from (1.17) and (1.18), the ‘vacuum’ expectation values of a monomial of degree n in Y_α is determined by the ‘vacuum’ expectation values of a monomial of degree $n - 1$ and one of degree $n - 2$ in Y_α . This implies the existence of recurrence relations between stationary distribution functions $P_s(\{\beta\})$ of lattices of different length L (which is also the degree of the monomial in question). The first calculation of correlation functions in this model were performed using these recurrence relations without using the algebraic approach [2, 3].

One may try to take more bulk rates than the purely diffusive or hopping ones just described. The problem is that (1.12) with X_α ’s chosen to be c -numbers gives three equations for the two generators D_0 and D_1 , while one is enough to determine all ‘vacuum’ expectation values as we have just seen. This implies relations between the rates. One trivial case corresponds to one-dimensional representations for the D_α (see [9]) when the (connected) correlation functions vanish. Even two-dimensional representations do not exist, since the relations on the rates obtained are incompatible with their positivity [16]. Another possibility is to not restrict the X_α ’s to be c -numbers, but (say) to take

$$X_\alpha = A_{\alpha\gamma} D_\gamma + x_\alpha \quad (1.19)$$

where $A_{\alpha\gamma}$ is an arbitrary 2×2 matrix and x_α are arbitrary c -numbers. Using the arbitrariness of the matrix elements $A_{\alpha\gamma}$, Krebs *et al* [16] have shown that it is possible to include more non-zero rates than just the hopping ones (Γ_{10}^{01} and Γ_{01}^{10}) and get a two-dimensional representation of the quadratic algebra, and hence obtain non-trivial correlation functions.

We would like to mention that the algebraic approach can be used not only in sequential processes described by the master equation, but also for parallel dynamics [17–19], where the algebraic problems to be solved are identical. Also, it is well worth recalling that matrix-product states first appeared in the literature for periodic boundary conditions in the work of Hakim and Nadal [20], and have also appeared in various publications by Fannes *et al* [21] who also considered Bethe lattices. Parallel work was done by the Köln group [22]. The algebraic problem for periodic boundary conditions is very different since, in this case, the wavefunction is given in terms of traces of monomials and not in terms of inner products in Fock states. Quadratic algebras can be used in this case too (see [23–25]). Last but not least, we would like to mention that quadratic algebras were also used for non-stationary properties like diffusion constants [26,27]. A review of some of the early work on the subject can be found in [28].

In the present paper, we address the problem of stochastic processes involving $N \geq 2$ species. We shall consider not only the case of open boundary conditions, but also, for the first time, closed or mixed boundary conditions since the algebraic approach applies to these cases as well. In section 2, we first review some properties of intensity matrices which we shall use in the following sections. In section 3, we consider the problem of the linear chain with closed ends (free boundary conditions). The boundary matrices R and L are identically zero and we shall consider solutions where $X_\alpha = 0$. This case is not only interesting on its own, but it represents the natural first step before considering the chain with open ends where R and L are non-zero. (The choice $X_\alpha = 0$ is compatible with equation (1.13) and will take us to an interesting class of quadratic algebras.) This implies

$$\Gamma_{\gamma\delta}^{\alpha\beta} D_\alpha D_\beta = 0. \tag{1.20}$$

We shall consider a quadratic algebra which contains only quadratic terms like in equation (1.20) a *polynomial algebra*. (Obviously the algebra defined by equation (1.17) is not one.) Equation (1.20) describes relations among monomials of degree two. The number of relations depends on the rank of the bulk intensity matrix and, using equation (1.14), we get the probability distribution for the two-sites case. In order to have a solution for more sites, one needs consistency conditions which imply that the two ways of relating the cubic monomials $D_\alpha D_\beta D_\gamma$ and $D_\gamma D_\beta D_\alpha$

$$\begin{array}{ccc}
 & D_\alpha D_\gamma D_\beta \rightarrow D_\gamma D_\alpha D_\beta & \\
 D_\alpha D_\beta D_\gamma & \nearrow & \searrow \\
 & D_\beta D_\alpha D_\gamma \rightarrow D_\gamma D_\beta D_\alpha & \\
 & \searrow & \nearrow \\
 & & D_\gamma D_\beta D_\alpha
 \end{array} \tag{1.21}$$

give the same result. The requirement of commutativity of the above diagram imposes constraints on the bulk rates and solves the $L = 3$ case. In all the examples presented in this paper, we have checked that, once the relations (1.21) are satisfied, no supplementary conditions on the rates arise from quartic (and in some cases) higher degree monomials. Similar conditions appear in quantum group structures [29–31] and it is no accident that various quantum planes or superplanes are solutions of the consistency conditions in some of the cases that we consider. Let us observe that for the closed chain case one does not need to take average values $\langle 0 | \dots | 0 \rangle$ in equation (1.14); the ground-state wavefunctions (the ground state is often degenerate) and therefore the probability distributions are the *coefficients of the independent monomials*. Let us observe that the early applications [20–22] of matrix-product states for periodic boundary conditions, in which the Hamiltonian was not given by intensity matrices, have used polynomial algebras (1.20) which have representations with finite traces. The existence of finite traces implies supplementary conditions on the $\Gamma_{\gamma,\delta}^{\alpha,\beta}$

in equation (1.20) besides the associativity conditions (1.21) [25]. In section 3 we also answer the following question: If the ground state can be obtained by the present algebraic methods, is the Hamiltonian integrable? We shall show that the answer is, in general, no.

In section 4, we consider the case where the left end of the linear chain is open ($L_\beta^\alpha \neq 0$) and the right end is closed (R_β^α). We again take $X_\alpha = 0$ but now we need to add the condition

$$L_\alpha^\beta \langle 0 | D_\beta = 0 \quad (\alpha = 0, 1, \dots, N-1) \quad (1.22)$$

to equation (1.20), which comes from equation (1.13). This implies new consistency conditions between the bulk and the left-boundary intensity matrices, which come from the equations

$$L_\alpha^\beta \langle 0 | D_\beta D_\gamma = 0. \quad (1.23)$$

The consistency conditions depend on the rank of the L matrix. Note that in this case one has to drop the $|0\rangle$ symbol and retain the $\langle 0|$ symbol in equation (1.14), where the stationary probability distribution is expressed. Again there are no Fock representations in this case.

Fock representations appear in section 5, where we consider the linear chain with both ends open. This problem is much more difficult than the preceding ones. We, therefore, limit ourselves to the case $N = 3$ only, since this case is complex enough and significantly different from the $N = 2$ case. We also restrict our attention to the case of diffusion only (in the bulk), i.e. the only non-zero bulk rates are $\Gamma_{\beta\alpha}^{\alpha\beta}$. We look for representations for the case where the X_α 's are c -numbers. To our knowledge, only one example of this kind was known up to now [5]. It is possible to perform a linear transformation to bring (1.12) and (1.13) into the form

$$[Y_\alpha, Y_\beta] = \sum_{\gamma, \delta \in \{0, 1, \dots, N-1\}} c_{\alpha\beta}^{\gamma\delta} Y_\gamma Y_\delta + \sum_{\gamma \in \{0, 1, \dots, N-1\}} c_{\alpha\beta}^\gamma Y_\gamma + c_{\alpha\beta} \quad (1.24)$$

where

$$c_{\alpha\beta}^{\gamma\delta} = c_{\alpha\beta}^{\delta\gamma} = -c_{\beta\alpha}^{\gamma\delta} \quad c_{\alpha\beta}^\gamma = -c_{\beta\alpha}^\gamma \quad c_{\alpha\beta} = -c_{\beta\alpha} \quad (1.25)$$

with $N = 3$, and the Fock conditions derived from equation (1.13). We review the properties of the algebra (1.24) and stress that their properties are essentially different if $N > 2$. We also show that the Fock conditions for $N > 2$ are in general too numerous. This leads to a careful separation of cases depending on the rank of the boundary intensity matrices.

In section 6, we show that if all the minors of the boundary matrices are non-zero, one can have representations of the quadratic algebra of dimension at most two with the bulk and boundary rates lying on some algebraic variety. In section 7, we consider the case in which only one minor of L_β^α and one of R_β^α are non-zero. We give the representations of the algebra in this case. Other cases (three or more minors non-zero, for example) are not presented since the paper is long enough even without them. In the appendix, we present simple physical processes where the formalism may be applied. (The results in the different sections of the paper may be used for several other applications.) We consider the problem of spontaneous symmetry breaking in a two-species exclusion model with asymmetric diffusion proposed by Evans *et al* [5]. More boundary and bulk rates are considered than in the original model. This extension might allow a better understanding of the physics of the problem.

In section 8, we consider the case in which all the principal cofactors are zero (the boundary intensity matrices have rank one). We are going to give four examples for this case.

This is a difficult paper to read since there is no physical intuition which facilitates the anticipation of, or allows us to guess, the results obtained. We shall permit ourselves to

advise the reader on how to get through the paper. If one is interested only in quadratic algebras, examples of polynomial algebras (1.20) are found in section 3. Several examples of more general quadratic algebras (1.24) are found in section 7. If one is interested in physical applications, one should start with section 2 and decide what type of physical problems one is interested in. If one looks to problems with one or two closed ends, sections 3 and 4 are enough. If one is interested in problems with open ends, the reader should know that we have considered only the case where the number of particles is conserved in the bulk. Next she/he should know which type of boundary intensity matrices are dictated by the physical problem and compute their rank. This is a crucial point. If the boundary intensity matrices have non-zero minors, she/he should consult section 6 and be aware that the present method is unlikely to be of any help. If, however, some or all the cofactors are zero, the methods presented in this paper might be very useful.

The reader who might be tired of this long introduction is invited to go directly to section 9, where a guide to finding the new results is presented along with a number of open questions.

2. Some properties of intensity matrices

An $N \times N$ matrix M whose elements M_{ij} are such that the sum of all its entries on each column vanishes, i.e.

$$\sum_{i=0}^{N-1} M_{ij} = 0 \quad (j = 0, 1, \dots, N - 1) \tag{2.1}$$

is called an *I-matrix*. I-matrices are closed under multiplication and form an algebra.

For stochastic problems, one considers I-matrices which have to satisfy the further restriction that the off-diagonal elements are real and non-negative, since they are interpreted as probability rates of certain processes. Such an $N \times N$ matrix M with real entries M_{ij} is called an *intensity matrix* if

- (i) all its off-diagonal entries are non-negative, $M_{ij} \geq 0, i \neq j$;
- (ii) and the diagonal elements are negative with

$$M_{ii} = - \sum_{i \neq j} M_{ij}. \tag{2.2}$$

For any matrix A with entries A_{ij} , if the Laplace row expansion of a determinant is written as $\sum_j A_{ij} \mathcal{A}_{kj} = \delta_{ik} \det A$ where \mathcal{A}_{ij} is the *cofactor* corresponding to the (i, j) th element of A . An important property of any zero column-sum matrix is that the cofactors for each column are equal in magnitude. (This can be proved as follows. The difference of any two successive cofactors down any column of a $n \times n$ matrix A , $\mathcal{A}_{i,j} - \mathcal{A}_{i+1,j}$, is a sum of determinants of two $(n - 1) \times (n - 1)$ matrices which differ only in one row, which can thus be expressed as the determinant of the sum of these matrices which is a zero column-sum matrix.) In the case of intensity matrices, all the cofactors have the same sign. Let us denote the column- j cofactor of an intensity matrix M by the corresponding symbol in calligraphic font \mathcal{M}_j .

It, therefore, follows that a system of linear equations

$$\sum_{n=0}^{N-1} M_{mn} x_n = 0 \tag{2.3}$$

has a solution given by

$$x_n = \xi \mathcal{M}_n \tag{2.4}$$

for any constant (independent of n) ξ if the rank of M is $N - 1$.

Consider two intensity matrices F and G , and their sum

$$E = F + G \quad (2.5)$$

which is also an intensity matrix. The following identity is true for $N = 1, 2$ and 3 only:

$$\sum_{m=0}^{N-1} (F_{nm} - G_{nm}) \mathcal{E}_m = (N - 1) \sum_{m=0}^{N-1} (F_{nm} \mathcal{G}_m - G_{nm} \mathcal{F}_m) \quad (2.6)$$

where as noted earlier, the calligraphic \mathcal{E} , \mathcal{F} , \mathcal{G} with subscript m denotes the cofactors of E , F , G for the m th column. The last identity is useful for the three-species case. The identity (2.6) which can be easily proved is probably new. Since we are going to use them often in the next few sections, we give the expressions for the three distinct cofactors of a 3×3 intensity matrix F with matrix elements F_n^m :

$$\begin{aligned} \mathcal{F}_0 &= F_0^1 F_1^2 + F_0^2 F_1^0 + F_0^2 F_2^1 \\ \mathcal{F}_1 &= F_1^0 F_0^2 + F_1^2 F_1^0 + F_1^2 F_2^0 \\ \mathcal{F}_2 &= F_2^0 F_0^1 + F_2^1 F_2^0 + F_2^1 F_1^0. \end{aligned} \quad (2.7)$$

3. Steady states for a linear chain with closed ends (ground states for quantum spin chains with free boundary conditions)

As mentioned in the introduction, we consider ground-state wavefunctions of the form

$$|P_S\rangle\rangle = \prod_{k=1}^L \left(\sum_{\mu_k=0}^{N-1} D_{\mu_k} u_{\mu_k} \right) \quad (3.1)$$

where the matrices D_μ satisfy equations (1.20) and the consistency conditions (1.21). It is important to note that unlike equation (1.15) we do not use the matrix elements (no $\langle 0|$ or $|0\rangle$ symbols are necessary). The coefficients of the independent words (independent monomials in the D matrices) give the different ground states. If the ground state is not degenerate one has only one independent monomial. The type of wavefunctions we get depends on the rank of the bulk intensity matrix $\Gamma_{\gamma\delta}^{\alpha\beta}$. Let us first consider the case where the rank is the maximum possible, i.e. $N^2 - 1$, and all principal minors are non-zero. We denote by $\mathcal{G}_{\alpha\beta}$ the cofactors of $\Gamma_{\gamma\delta}^{\alpha\beta}$ (see section 2). Using equation (2.4) we get

$$D_\alpha D_\beta = \frac{\mathcal{G}_{\alpha\beta}}{\mathcal{G}_{\gamma\delta}} D_\gamma D_\delta. \quad (3.2)$$

There are two ways in which the cubic monomial $D_\alpha D_\beta D_\gamma$ can be related to $D_\mu D_\rho D_\sigma$:

$$D_\alpha D_\beta D_\gamma = \frac{\mathcal{G}_{\alpha\beta}}{\mathcal{G}_{\mu\nu}} D_\mu D_\nu D_\gamma = \frac{\mathcal{G}_{\alpha\beta}}{\mathcal{G}_{\mu\nu}} \frac{\mathcal{G}_{\nu\gamma}}{\mathcal{G}_{\rho\sigma}} D_\mu D_\rho D_\sigma \quad (3.3a)$$

$$D_\alpha D_\beta D_\gamma = \frac{\mathcal{G}_{\beta\gamma}}{\mathcal{G}_{\nu\sigma}} D_\alpha D_\nu D_\sigma = \frac{\mathcal{G}_{\beta\gamma}}{\mathcal{G}_{\nu\sigma}} \frac{\mathcal{G}_{\alpha\nu}}{\mathcal{G}_{\mu\rho}} D_\mu D_\rho D_\sigma. \quad (3.3b)$$

Comparing equations (3.3a) and (3.3b) (and setting $\rho = \nu$) we get

$$\mathcal{G}_{\alpha\beta} \mathcal{G}_{\nu\gamma} = \mathcal{G}_{\alpha\nu} \mathcal{G}_{\beta\gamma} \quad (3.4)$$

and hence the condition on the cofactors (on setting $\nu = \alpha$ and $\beta = \gamma$) is

$$\mathcal{G}_{\alpha\beta} = \pm \sqrt{\mathcal{G}_{\alpha\alpha} \mathcal{G}_{\beta\beta}} = \mathcal{G}_{\beta\alpha}. \quad (3.5)$$

The $+(-)$ is taken if N is odd (even). This implies that D_α can be taken as c -numbers:

$$D_\alpha = \xi \sqrt{G_{\alpha\alpha}} \tag{3.6}$$

where ξ is a constant. The ground state is unique and the (connected) correlation functions manifestly zero.

The whole structure of the possible solutions for the ground-state wavefunction depends on the rank of the bulk intensity matrix $\Gamma_{\gamma\delta}^{\alpha\beta}$. From now on we shall restrict ourselves to the cases $N = 2, 3$ only.

In what follows, we shall be able to find many examples of ground-state wavefunctions using polynomial algebras. Does this imply the full integrability of the Hamiltonian? One could be tempted to think so. As we are going to see, the answer in general is negative. Exact integrability requires more than an analytical expression for the ground-state wavefunctions. In order to clarify this point it is useful to remind the reader of some mathematical tools.

In certain cases, the e_k defined by

$$e_k := H_{k,k+1} \tag{3.7}$$

where $H_{k,k+1}$ is the Hamiltonian density given in equation (1.9), satisfy the relations of a Hecke algebra:

$$\begin{aligned} e_k e_{k\pm 1} e_k - e_k &= e_{k\pm 1} e_k e_{k\pm 1} - e_{k\pm 1} \\ [e_k, e_j] &= 0 \quad \text{for } |k - j| \geq 2 \\ e_k^2 &= (q + q^{-1})e_k. \end{aligned} \tag{3.8}$$

In this case, one can find a spectral parameter dependent solution $\check{R}(u)$ of the Yang–Baxter equation via ‘Baxterization’ [32, 33]. One can check that the generators e_k satisfy further relations among themselves which define various quotients of the Hecke algebra [34]. For our purposes, it is sufficient to state that the $H_{k,k+1}$ belong to such quotients that are classified by a pair of natural numbers (P, M) . For each quotient labelled by (P, M) the Hamiltonian whose density is given by the appropriate e_k defines a spin chain [35] with $U_q su(P|M)$ (which is the Schur–Weyl dual of the (P, M) Hecke quotient) as its symmetry algebra. (For further details, see [11].)

A more general criterion for integrability at the level of Hamiltonian densities was introduced by Reshetikhin [36] for the case of R -matrices whose dependence on the spectral parameter is of difference type. By expanding the R and row transfer matrices in powers of the spectral parameter, it was shown to be necessary that the Hamiltonian densities e_k satisfy the following relation

$$[e_k + e_{k+1}, [e_k, e_{k+1}]] = W_k - W_{k+1} \tag{3.9}$$

where W_k is a tensor product of identity matrices at all sites on the chain and an arbitrary matrix at the k th site. We use these criteria as tests for integrability for a few examples.

• $N = 2$. We now consider the $N = 2$ case. We shall on occasion refer to the state ‘1’ as that denoting the presence of a particle and the state ‘0’ to its absence. If all the minors are non-zero, and the intensity matrix is of maximal possible rank (i.e. 3) equation (3.6) applies. We consider the cases where the rank is less than 3. Let us first consider the case

$$\Gamma_{10}^{00} = \Gamma_{01}^{00} = \Gamma_{11}^{00} = \Gamma_{00}^{01} = \Gamma_{00}^{10} = \Gamma_{00}^{11} = 0 \tag{3.10}$$

where the bulk intensity matrix effectively reduces (from a 4×4) to a 3×3 matrix with rows and columns labelled by $(1,0)$, $(0,1)$ and $(1,1)$. Let $\mathcal{G}'_{1,0}$, $\mathcal{G}'_{0,1}$ and $\mathcal{G}'_{1,1}$ be the cofactors of this new intensity matrix. In order to have a polynomial algebra we must have the condition

$$\mathcal{G}'_{1,0} = \mathcal{G}'_{0,1} \tag{3.11}$$

and the algebra is

$$D_1 D_0 = D_0 D_1 = \frac{\mathcal{G}'_{1,0}}{\mathcal{G}'_{1,1}} D_1^2. \tag{3.12}$$

The ground state for L sites is now doubly degenerate. One state corresponds to D_0^L and the other contains all the monomials which have at least one D_1 , and can be brought by equation (3.12) to be proportional to D_1^L .

Next we consider the situation where we have only diffusion of particles and their annihilation if two particles are on neighbouring sites. This is attained by choosing the following rates to vanish (the rest being non-zero):

$$\Gamma_{10}^{00} = \Gamma_{01}^{00} = \Gamma_{11}^{00} = \Gamma_{00}^{01} = \Gamma_{00}^{10} = \Gamma_{11}^{01} = \Gamma_{11}^{10} = 0. \tag{3.13}$$

In this case the algebra is

$$D_0 D_1 = q^2 D_1 D_0 \quad D_1^2 = 0 \tag{3.14}$$

where

$$q = \sqrt{\frac{\Gamma_{01}^{10}}{\Gamma_{10}^{01}}}. \tag{3.15}$$

If we choose the time scale such that $\Gamma_{01}^{10} \Gamma_{10}^{01} = 1$, the set of $H_{k,k+1}$ satisfies the relations for the (1,1) quotient of the Hecke algebras centralizing $U_q(su(1|1))$. The ground state for L sites is doubly degenerate corresponding to the words D_0^L and $D_1 D_0^{L-1}$.

Yet another example of an intensity matrix which gives rise to a quadratic algebra is

$$\begin{pmatrix} -\Gamma_{10}^{00} & 0 & \Gamma_{00}^{10} & 0 \\ 0 & -\Gamma_{11}^{01} & 0 & \Gamma_{01}^{11} \\ \Gamma_{10}^{00} & 0 & -\Gamma_{00}^{10} & 0 \\ 0 & \Gamma_{11}^{01} & 0 & -\Gamma_{01}^{11} \end{pmatrix} \tag{3.16}$$

which gives the following quadratic algebra:

$$D_0 D_1 = \frac{1}{q^2} \quad \text{and} \quad D_1 D_0 = q^2 D_0^2$$

where $q^2 = \frac{\Gamma_{10}^{00}}{\Gamma_{00}^{10}} = \frac{\Gamma_{11}^{01}}{\Gamma_{01}^{11}}$ (3.17)

after solving for the associativity constraints coming from the cubic terms.

Yet another possibility is to set more rates to zero, in addition to those which led to equation (3.14). Upon setting Γ_{00}^{11} , Γ_{01}^{11} and Γ_{10}^{11} to zero as well, which means we are only considering diffusion, we are led to

$$D_0 D_1 = \frac{\Gamma_{01}^{10}}{\Gamma_{10}^{01}} D_1 D_0. \tag{3.18}$$

The ground state is now $(L + 1)$ times degenerate—each wavefunction has a given number of D_1 's and the coefficients of the tensor products of $u_k^{(\mu)}$ in equation (3.1) correspond to the q -deformation of the symmetrizer corresponding to the Young diagram with one row and L boxes [11], with $q = \sqrt{\Gamma_{01}^{10} / \Gamma_{10}^{01}}$. The quadratic relation

$$D_0 D_1 = 0 \tag{3.19}$$

corresponds not only to the case where Γ_{10}^{01} is the only non-zero rate, but more generally $\Gamma_{\alpha\beta}^{01}$ is non-zero for all α, β . Again the ground state is $(L + 1)$ times degenerate. Since

our intention is to generalize the problem to $N > 2$, it is useful to present visually the non-zero matrix elements which correspond to the different quadratic relations. The rows and columns below are labelled (0, 0), (0, 1), (1, 0) and (1, 1) consecutively.

$$\begin{pmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{pmatrix} \text{ equation (3.6)} \qquad \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \square & \square & \square \\ \cdot & \square & \square & \square \\ \cdot & \square & \square & \square \end{pmatrix} \text{ equation (3.12)}$$

$$\begin{pmatrix} \cdot & \cdot & \cdot & \square \\ \cdot & \square & \square & \square \\ \cdot & \square & \square & \square \\ \cdot & \cdot & \cdot & \square \end{pmatrix} \text{ equation (3.14)} \qquad \begin{pmatrix} \square & \cdot & \square & \cdot \\ \cdot & \square & \cdot & \square \\ \square & \cdot & \square & \cdot \\ \cdot & \square & \cdot & \square \end{pmatrix} \text{ equation (3.17)}$$

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \square & \square & \cdot \\ \cdot & \square & \square & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \text{ equation (3.18)} \qquad \begin{pmatrix} \cdot & \square & \cdot & \cdot \\ \cdot & \square & \cdot & \cdot \\ \cdot & \square & \cdot & \cdot \\ \cdot & \square & \cdot & \cdot \end{pmatrix} \text{ equation (3.19)}.$$

We could not obtain algebras with non-scalar D_0 and D_1 for any other choice of locations in the intensity matrix for the non-zero rates than those depicted above.

• $N = 3$. We now consider the $N = 3$ case. We are not going to list all the possible quadratic relations for the two-site problem which are compatible with the three-site problem (i.e. the cubic relations are compatible with the associative application of the quadratic relations) but list a few examples which do. We shall, in some cases, interpret the state variables 1, 2, 0 as those of two species of particles and holes, respectively. First, we consider the case in which the two-site bulk intensity matrix decouples into three intensity matrices (the other rates being zero):

$$\begin{pmatrix} -\Gamma_{10}^{01} & \Gamma_{01}^{10} \\ \Gamma_{10}^{01} & -\Gamma_{01}^{10} \end{pmatrix} \qquad \begin{pmatrix} -\Gamma_{20}^{02} & \Gamma_{02}^{20} \\ \Gamma_{20}^{02} & -\Gamma_{02}^{20} \end{pmatrix}$$

and

$$\begin{pmatrix} -(\Gamma_{21}^{00} + \Gamma_{12}^{00}) & \Gamma_{00}^{12} & \Gamma_{00}^{21} \\ \Gamma_{12}^{00} & -(\Gamma_{00}^{12} + \Gamma_{21}^{12}) & \Gamma_{12}^{21} \\ \Gamma_{21}^{00} & \Gamma_{21}^{12} & -(\Gamma_{00}^{21} + \Gamma_{12}^{21}) \end{pmatrix}. \tag{3.20}$$

We denote by \mathcal{I}_{00} , \mathcal{I}_{12} and \mathcal{I}_{21} the cofactors of the 3×3 intensity matrix. The quadratic relations obtained are

$$\begin{aligned} D_0 D_1 &= \gamma D_1 D_0 & D_2 D_0 &= \delta D_0 D_2 \\ D_1 D_2 &= \beta D_0^2 & \alpha D_2 D_1 &= \beta D_0^2. \end{aligned} \tag{3.21}$$

The cubic relations obtained by considering words in the D 's of degree three are compatible with the quadratic relations provided

$$\alpha = \frac{\mathcal{I}_{12}}{\mathcal{I}_{21}} = \gamma^{-2} \qquad \beta = \frac{\mathcal{I}_{12}}{\mathcal{I}_{00}} \qquad \gamma = \delta = \frac{\Gamma_{01}^{10}}{\Gamma_{10}^{01}} = \frac{\Gamma_{20}^{02}}{\Gamma_{02}^{20}}. \tag{3.22}$$

There are $(2L + 1)$ wavefunctions with energy eigenvalue zero. They are given in terms of the monomials

$$D_0^{L-m} D_1^m \qquad D_0^{L-n} D_2^n. \tag{3.23}$$

This gives an explanation as to why it was possible to calculate the probability distribution exactly in [37] for the particular case where interchange of particles is forbidden ($\Gamma_{21}^{12} = \Gamma_{12}^{21} = 0$). Interpreting the matrix entries in equation (3.20) as transition rates involving

two species of particles 1 and 2 we notice that the difference in the number of particles of types 1 and 2 is conserved under this stochastic dynamics.

We now consider the intensity matrix of the block diagonal form

$$\begin{pmatrix} \Gamma_{00}^{00} & \Gamma_{00}^{01} & \Gamma_{00}^{02} & 0 & 0 & 0 & 0 & 0 & 0 \\ \Gamma_{01}^{00} & \Gamma_{01}^{01} & \Gamma_{01}^{02} & 0 & 0 & 0 & 0 & 0 & 0 \\ \Gamma_{02}^{00} & \Gamma_{02}^{01} & \Gamma_{02}^{02} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Gamma_{10}^{10} & \Gamma_{10}^{11} & \Gamma_{10}^{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Gamma_{11}^{10} & \Gamma_{11}^{11} & \Gamma_{11}^{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Gamma_{12}^{10} & \Gamma_{12}^{11} & \Gamma_{12}^{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Gamma_{20}^{20} & \Gamma_{20}^{21} & \Gamma_{20}^{22} \\ 0 & 0 & 0 & 0 & 0 & 0 & \Gamma_{21}^{20} & \Gamma_{21}^{21} & \Gamma_{21}^{22} \\ 0 & 0 & 0 & 0 & 0 & 0 & \Gamma_{22}^{20} & \Gamma_{22}^{21} & \Gamma_{22}^{22} \end{pmatrix} \quad (3.24)$$

and we denote the cofactors of the 3×3 block containing Γ_{ii}^{ii} by $\mathcal{G}_{(i)}^\alpha$, where $\alpha = 0, 1, 2$ labels the $(\alpha + 1)$ th column within each block. The algebra is defined by the six relations

$$D_\alpha D_\beta = \frac{f_\beta}{f_\alpha} D_\alpha^2 \quad (3.25)$$

where

$$\frac{f_1}{f_0} = \frac{\mathcal{G}_{(0)}^1}{\mathcal{G}_{(0)}^0} = \frac{\mathcal{G}_{(1)}^1}{\mathcal{G}_{(1)}^0} \quad \frac{f_0}{f_2} = \frac{\mathcal{G}_{(2)}^0}{\mathcal{G}_{(2)}^1} = \frac{\mathcal{G}_{(0)}^0}{\mathcal{G}_{(2)}^0} \quad (3.26)$$

$$\frac{f_2}{f_1} = \frac{\mathcal{G}_{(1)}^2}{\mathcal{G}_{(1)}^1} = \frac{\mathcal{G}_{(2)}^2}{\mathcal{G}_{(2)}^1} \quad \mathcal{G}_{(0)}^0 \mathcal{G}_{(1)}^1 \mathcal{G}_{(2)}^2 = \mathcal{G}_{(0)}^1 \mathcal{G}_{(1)}^2 \mathcal{G}_{(2)}^0. \quad (3.27)$$

The ground state is threefold degenerate, the independent words being D_0^L , D_1^L and D_2^L . This degeneracy arises since the type of particle located at the first site $(0, 1, 2)$ is unchanged by the stochastic dynamics.

An interesting example with four quadratic relations is

$$\begin{aligned} \Gamma_{10}^{01} D_0 D_1 &= \Gamma_{01}^{10} D_1 D_0 \\ \Gamma_{21}^{12} D_1 D_2 &= \Gamma_{12}^{21} D_2 D_1 \\ \Gamma_{02}^{20} D_2 D_0 &= \Gamma_{20}^{02} D_0 D_2 \\ D_1^2 &= 0. \end{aligned} \quad (3.28)$$

These relations are obtained if the only non-zero rates are those that appear in (3.28) and $\Gamma_{\alpha\beta}^{11} (\alpha, \beta \in \{0, 1\})$. In order to find the degeneracy of the ground state, notice that if we set

$$\Gamma_{\beta\alpha}^{\alpha\beta} = \frac{1}{\Gamma_{\alpha\beta}^{\beta\alpha}} \quad (\alpha \neq \beta) \quad \text{and} \quad \frac{\Gamma_{10}^{01}}{\Gamma_{01}^{10}} = \frac{\Gamma_{20}^{02}}{\Gamma_{02}^{20}} = \frac{\Gamma_{21}^{12}}{\Gamma_{12}^{21}} = q^2 \quad (3.29)$$

we obtain the $(2, 1)$ quotient of the Hecke algebra [11] and, by Schur–Weyl duality, the Hamiltonian is $U_q(su(2|1))$ symmetric. The ground-state wavefunctions are proportional to the monomials symmetrized using the relations (3.28) and (3.29) and are, therefore, given by the representation of the superalgebra which corresponds to a Young diagram with L boxes in one row [38].

A physically important example is the case in which we consider hopping and interchange of particles; we set the rates $\Gamma_{\alpha\beta}^{11}$ to zero in the previous example to get the three relations

$$D_0 D_1 = \frac{\Gamma_{01}^{10}}{\Gamma_{10}^{01}} D_1 D_0 \quad D_1 D_2 = \frac{\Gamma_{12}^{21}}{\Gamma_{21}^{12}} D_2 D_1 \quad D_2 D_0 = \frac{\Gamma_{20}^{02}}{\Gamma_{02}^{20}} D_0 D_2. \quad (3.30)$$

We have checked whether the Hamiltonian thus defined satisfies the necessary condition as prescribed by Reshetikhin (equation (3.9)) for being integrable. We found that in order to be so, the rates must satisfy the relations (3.29) and we get the (3, 0) quotient of the Hecke algebra and the Hamiltonian is $U_q(su(3))$ symmetric. This result is important because it shows that, although the ground state of the Hamiltonian can be obtained by algebraic means, the spectrum of the Hamiltonian cannot, unless supplementary conditions are satisfied. We have also checked other cases with the same result. If the relations (3.29) are satisfied, the ground states again give the representation of $U_q(su(3))$ which corresponds to the Young diagram with L boxes in one row [38].

We can also consider the case where the only rates that are non-zero are Γ_{10}^{01} , Γ_{01}^{10} , Γ_{20}^{02} and Γ_{02}^{20} . The only two relations that survive are

$$\Gamma_{10}^{01} D_0 D_1 = \Gamma_{01}^{10} D_1 D_0 \quad \text{and} \quad \Gamma_{20}^{02} D_0 D_2 = \Gamma_{02}^{20} D_2 D_0 \quad (3.31)$$

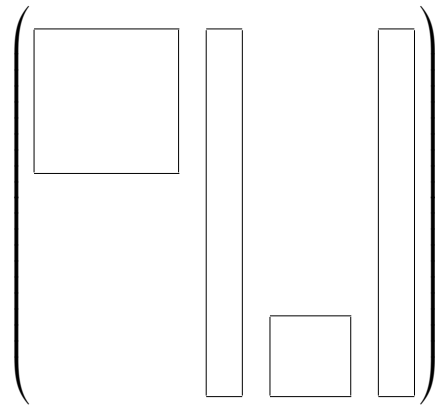
and the degeneracies are huge since any sequence of the state indices 1 and 2 is invariant under the dynamics and can be computed using the results of [39].

Finally, one can have only one relation

$$D_i D_j = 0 \quad (3.32)$$

if the only non-zero rates are $\Gamma_{\alpha,\beta}^{i,j}$ for any α, β .

Since we have only looked at particular cases, it is most likely that we have not exhausted all possible quadratic algebras arising from 9×9 intensity matrices. We have not been able to find other examples which give non-trivial (non-scalar) representations for the relations emerging from a choice of intensity matrix. The rules of the game are, however, very simple and can be applied by the reader to find other examples. In the 9×9 intensity matrix, one takes several diagonal blocks of smaller intensity matrices with non-zero rates and columns of possibly non-zero rates of the following form:



Notice that there are no rectangular boxes which span entire rows of the matrix Γ above.

This intensity matrix Γ gives quadratic relations which then have to be checked for consistency for the three-site case. We can do the same with another intensity matrix $\Gamma' := P_{\mu\nu} \Gamma P_{\mu\nu}$, where $P_{\mu\nu}$ is a permutation matrix,

$$P_{\mu\nu}(u_{\mu_k} \otimes u_{\nu_{k+1}}) = (u_{\nu_k} \otimes u_{\mu_{k+1}})$$

where u_{μ_k, ν_k} are defined before equation (1.15). The same procedure applies for any number of species N .

To sum up, we have shown that there are a number of quadratic algebras which allow the computation of the ground-state wavefunction. What is missing is a complete classification

of the algebras. What is also missing is an understanding of the origin of the degeneracies. Are the ground-state wavefunctions some representations of some algebra? In the special case where the Hamiltonian densities are generators of a Hecke algebra, the answer is known, but now in the general case.

We would also like to emphasize that the ground-state wavefunctions thus computed are non-trivial in the sense that connected correlations functions in these states are non-zero. One simple example is already known. This is given in equation (3.18) [40]. The method used in [40] to compute the correlation functions was different. Other cases can be considered using the results of this section.

4. Linear chain with left end open and right end closed

As discussed in section 1, the ground-state wavefunction is given by the expression

$$|P_s\rangle\rangle = \langle 0| \prod_{k=1}^L \left(\sum_{\mu_k=0}^{N-1} D_{\mu_k} u_{\mu_k} \right) \quad (4.1)$$

where D_μ satisfy one of the polynomial algebras described in the last section and a new condition (see equation (1.22))

$$\sum_{\beta=0}^{N-1} L_\alpha^\beta \langle 0| D_\beta = 0 \quad (4.2)$$

where L_α^β is an intensity matrix with cofactors \mathcal{L}_β . Notice that as compared to equation (1.15) we use only the bra $\langle 0|$ symbol and no ket $|0\rangle$ symbol. If the ground state is degenerate the coefficients of the independent bra vectors (monomials of D applied to $\langle 0|$) will give the independent wavefunctions. If the ground state is not degenerate one has only one independent bra vector. We consider the cases $N = 2$ and $N = 3$ only. The following conditions on the D matrices are obtained depending on the structure of the boundary intensity matrix. In each case, only the non-zero rates are written down.

- $N = 2$

(i)

$$\langle 0| D_1 = \frac{L_1^0}{L_0^1} \langle 0| D_0 \quad (L_0^1, L_1^0 \neq 0) \quad (4.3)$$

(ii)

$$\langle 0| D_1 = 0 \quad (L_0^1 \neq 0) \quad (4.4)$$

(iii)

$$\langle 0| D_0 = 0 \quad (L_1^0 \neq 0). \quad (4.5)$$

- $N = 3$. For the case where at least one principal cofactor, for example \mathcal{L}_0 , is non-zero

$$\langle 0| D_\alpha = \frac{\mathcal{L}_\alpha}{\mathcal{L}_0} \langle 0| D_0 \quad (\alpha = 1, 2) \quad (4.6)$$

and one obtains two constraints. If all the principal cofactors are zero, only two rates are non-zero. We can distinguish the following cases:

(i)

$$\langle 0| D_1 = \frac{L_1^0}{L_0^1} \langle 0| D_0 \quad L_0^1, L_1^0 \neq 0 \quad (4.7)$$

(ii)

$$\langle 0|D_2 = \frac{L_2^0}{L_0^2} \langle 0|D_0 \quad L_0^2, L_2^0 \neq 0 \tag{4.8}$$

(iii)

$$\langle 0|D_2 = \frac{L_2^1}{L_1^2} \langle 0|D_1 \quad L_2^1, L_1^2 \neq 0 \tag{4.9}$$

(iv)

$$\langle 0|D_1 = 0 \quad (L_0^1 + L_2^1) \neq 0 \tag{4.10}$$

(v)

$$\langle 0|D_0 = 0 \quad (L_1^0 + L_2^0) \neq 0 \tag{4.11}$$

(vi)

$$\langle 0|D_2 = 0 \quad (L_0^2 + L_1^2) \neq 0. \tag{4.12}$$

Notice that for the last six cases one has only one constraint. The conditions (4.3)–(4.12) have to be compatible with the polynomial algebras. In this way one can, in some cases, find relations between the bulk rates only. In most cases the degeneracy of the ground state is lifted.

The marriage of the bulk algebra with the boundary conditions (4.3)–(4.12) is a straightforward mathematical exercise. Therefore, we shall confine ourselves to models of asymmetric diffusion where the only non-zero rates are $\Gamma_{\beta\alpha}^{\alpha\beta}$ and give examples of how the concentration profiles can be computed. These are the cases for which the relevant algebras are given in equation (3.18) for $N = 2$ and equation (3.30) for $N = 3$.

We shall start with the $N = 2$ case. We have the algebra

$$D_0D_1 = q^2D_1D_0 \tag{4.13}$$

and one of the conditions (4.3)–(4.5). The conditions (4.4) and (4.5) give a trivial ground state: $\langle 0|D_0^L$ (4.4) or $\langle 0|D_1^L$ (4.5). We shall rewrite equation (4.3) as follows:

$$\langle 0|D_1 = \mu \langle 0|D_0. \tag{4.14}$$

There are no constraints connecting q^2 and μ . Assume that D_1 indicates the presence of a particle on any particular site and D_0 the lack of one, i.e. holes. The concentration of particles at a distance k from the left boundary for a chain of length L is [4]

$$c(k) = \frac{1}{Z_L} \langle 0|C^{k-1}D_1C^{L-k} \tag{4.15}$$

where

$$Z_L = \langle 0|C^L \quad C = D_0 + D_1. \tag{4.16}$$

Obviously, equation (4.15) should be understood as the ratio of the coefficients of the same bra vector. A straightforward calculation gives

$$c(k) = \frac{1}{1 + q^{2(1-k)}/\mu}. \tag{4.17}$$

The connected two-point correlation function

$$c(k, l) = \frac{1}{Z_L} \langle 0|(C^{k-1}D_1C^{l-k-1}C_1C^{L-l}) - c(k)c(l) \tag{4.18}$$

vanishes.

An interesting case occurs for the algebra

$$D_1 D_0 = 0 \quad (4.19)$$

which corresponds to $\Gamma_{10}^{01} = 0$ where we consider the boundary condition $L_1^0 = 0$ (equation (4.4)). Upon inserting these conditions into equation (4.15) we obtain

$$c(k) = \frac{\langle 0 | \sum_{j=1}^{k-1} D_0^j D_1^{L-j} | 0 \rangle}{\langle 0 | \sum_{j=1}^L D_0^j D_1^{L-j} | 0 \rangle} \quad (4.20)$$

and we cannot go any further because our algebraic rules do not allow us to evaluate this expression. We leave it as an exercise to the reader to find the physical reasons why the method appears not to work in this case, and to do the calculation properly.

We now consider the $N = 3$ states problem and illustrate the method in the case of equation (3.30), which we rewrite for convenience

$$q_0 D_1 D_2 = D_2 D_1 \quad q_1 D_2 D_0 = D_0 D_2 \quad q_2 D_0 D_1 = D_1 D_0 \quad (4.21)$$

where

$$q_0 = \frac{\Gamma_{21}^{12}}{\Gamma_{12}^{21}} \quad q_1 = \frac{\Gamma_{02}^{20}}{\Gamma_{20}^{02}} \quad q_2 = \frac{\Gamma_{10}^{01}}{\Gamma_{01}^{10}}. \quad (4.22)$$

We consider the boundary conditions (4.6) and assume all the rates in (4.22) to be non-zero. Setting $\mu_\alpha := \mathcal{L}_\alpha / \mathcal{L}_0$, we have

$$\langle 0 | D_1 = \mu_1 \langle 0 | D_0 \quad \text{and} \quad \langle 0 | D_2 = \mu_2 \langle 0 | D_0. \quad (4.23)$$

As we shall now show, consistency conditions on the rates show up. Let us first compute

$$\begin{aligned} q_0 \langle 0 | D_1 D_2 &= q_0 \mu_1 \langle 0 | D_0 D_2 = q_0 q_1 \mu_1 \langle 0 | D_2 D_0 = q_0 q_1 \mu_1 \mu_2 \langle 0 | D_0^2 \\ \langle 0 | D_2 D_1 &= \mu_2 \langle 0 | D_0 D_1 = (\mu_2 / q_2) \langle 0 | D_1 D_0 = (\mu_1 \mu_2 / q_2) \langle 0 | D_0^2. \end{aligned} \quad (4.24)$$

From the first equation in (4.21) and equations (4.24) we get

$$q_0 q_1 q_2 = 1. \quad (4.25)$$

Note that this condition constrains only the bulk rates. The concentration profiles for particles '1' and '2' ('0' denotes vacancies) can easily be computed [5] as

$$\begin{aligned} c_1(k) &= \frac{1}{Z_L} \langle 0 | C^{k-1} D_1 C^{L-k} | 0 \rangle & c_2(k) &= \frac{1}{Z_L} \langle 0 | C^{k-1} D_2 C^{L-k} | 0 \rangle \\ Z_L &= \langle 0 | C^L | 0 \rangle & C &= D_0 + D_1 + D_2. \end{aligned} \quad (4.26)$$

Using equations (4.21), (4.23) and (4.25) we obtain

$$c_1(k) = \frac{\mu_1 q_2^{1-k}}{1 + \mu_1 q_2^{1-k} + \mu_2 q_1^{k-1}} \quad \frac{c_1(k)}{c_2(k)} = \frac{\mu_1}{\mu_2} q_0^{k-1}. \quad (4.27)$$

We are not aware of the existence of another method which would give us the expressions (4.27). The connected two-point correlation function is zero.

5. The open chain three-state exclusion models

This is a much more difficult problem. We limit our investigation to exclusion models in which the only non-vanishing rates in the bulk are

$$g_{\alpha\beta} := \Gamma_{\beta\alpha}^{\alpha\beta} \tag{5.1}$$

and make the assumption that the matrices X_μ in equation (1.12) are *c*-numbers, which we shall call x_μ . As we shall see below, the choice $x_\mu = 0$ is possible for pathological choices of boundary conditions only. The choice (5.1) is not only justified by the interest in these physical processes but, as shown later (see the discussion after equation (5.24)) in this way one obtained quadratic algebras which are understood [13, 14].

We are looking, therefore, at Fock representations of the algebra

$$\begin{aligned} g_{01}D_0D_1 - g_{10}D_1D_0 &= x_0D_1 - x_1D_0 \\ g_{12}D_1D_2 - g_{21}D_2D_1 &= x_1D_2 - x_2D_1 \\ g_{20}D_2D_0 - g_{02}D_0D_2 &= x_2D_0 - x_0D_2 \end{aligned} \tag{5.2}$$

with boundary conditions on the states $\langle 0|$ and $|0\rangle$

$$\langle 0|(x_\nu - L_\nu^\mu D_\mu) = 0 \quad \text{and} \quad (x_\nu + R_\nu^\mu D_\mu)|0\rangle = 0. \tag{5.3}$$

As shown in [9], for $N = 2$, one has Fock representations of the algebra for any bulk and boundary rates. The situation is going to be very different for $N = 3$. The reason is that the number of boundary conditions determined by the matrices L_ν^μ and R_ν^μ is, in general, too large. For each of L and R one has to consider all the cases enumerated in the previous section (see equations (4.6)–(4.12)). We now make some transformations which exploit the fact that the x_μ are still free parameters.

Let us first consider words of length 1:

$$\langle 0|D_\mu|0\rangle = \delta_\mu. \tag{5.4}$$

From equation (5.3), it follows that

$$(L_\nu^\mu + R_\nu^\mu)\delta_\mu = 0 \tag{5.5}$$

and

$$x_\nu = \frac{1}{2}(L_\nu^\mu - R_\nu^\mu)\delta_\mu. \tag{5.6}$$

We have to keep in mind that

$$B_\mu^\nu = L_\mu^\nu + R_\mu^\nu \tag{5.7}$$

is an intensity matrix with cofactors \mathcal{B}_μ . This implies (using equations (2.3)–(2.6)) that

$$\delta_\mu = \xi \mathcal{B}_\mu \tag{5.8}$$

and

$$x_\nu = \frac{1}{2}\xi(L_\nu^\mu - R_\nu^\mu)\mathcal{B}_\mu = \xi(L_\nu^\mu \mathcal{R}_\mu - R_\nu^\mu \mathcal{L}_\mu). \tag{5.9}$$

Here ξ is an arbitrary parameter and $\mathcal{R}_\mu(\mathcal{L}_\mu)$ are the cofactors of the intensity matrix $R_\nu^\mu(L_\nu^\mu)$. The expressions for the various cofactors can be obtained using equation (2.7). Since R_ν^μ and L_ν^μ are intensity matrices, it follows from equation (5.3) that

$$\sum_{\nu=0}^2 x_\nu = 0. \tag{5.10}$$

It is interesting to note that if we have a Fock representation, the ratio of the currents of particles 1 and 2 is already known. It is easy to show that

$$J_i = x_i \frac{\langle 0|C^{L-1}|0\rangle}{\langle 0|C^L|0\rangle} \quad i = 1, 2 \quad (5.11)$$

where

$$C = \sum_{\mu=0}^2 D_{\mu}. \quad (5.12)$$

This implies

$$\frac{J_1}{J_2} = \frac{x_1}{x_2} \quad (5.13)$$

where x_1/x_2 is given by the boundary conditions (see equation (5.9)).

In deriving (5.11) we have used the following expression for the current density [4, 5]

$$\begin{aligned} j_1 &= g_{10}D_1D_0 + g_{12}D_1D_2 - g_{01}D_0D_1 - g_{21}D_2D_1 \\ &= -(x_0 + x_2)D_1 + x_1(D_0 + D_2) = x_1C \end{aligned} \quad (5.14)$$

and a similar equation for j_2 . We also used the definition

$$J_i = \langle j_i \rangle. \quad (5.15)$$

It is easy to find the one-dimensional representations of the algebra (5.2). Using equation (5.4), we get the following conditions on the rates

$$\begin{aligned} g_{01} - g_{10} &= y_0 - y_1 \\ g_{12} - g_{21} &= y_1 - y_2 \\ g_{20} - g_{02} &= y_2 - y_0 \end{aligned} \quad (5.16)$$

where

$$y_{\mu} = \frac{x_{\mu}}{\delta_{\mu}} \quad (5.17)$$

are given by the boundary conditions (see equations (5.8) and (5.9)). Notice that equations (5.16) give one relation between the bulk rates,

$$g_{01} + g_{12} + g_{20} = g_{10} + g_{21} + g_{02} \quad (5.18)$$

and two relations between boundary and bulk rates.

Before we start looking for the cases in which the Fock representations of the algebra (5.2) exist, it is useful to bring it into a different form. We denote

$$u_0 = \frac{y_1 - y_2}{g_{21}} + 1 - q_0 \quad u_1 = \frac{y_2 - y_0}{g_{02}} + 1 - q_1 \quad u_2 = \frac{y_0 - y_1}{g_{10}} + 1 - q_2 \quad (5.19)$$

and also

$$\begin{aligned} v_{10} &= 1 - q_2 + (y_0/g_{10}) & v_{01} &= 1 - q_2 - (y_1/g_{10}) \\ v_{21} &= 1 - q_0 + (y_1/g_{21}) & v_{12} &= 1 - q_0 - (y_2/g_{21}) \\ v_{02} &= 1 - q_1 + (y_2/g_{02}) & v_{20} &= 1 - q_1 - (y_0/g_{02}). \end{aligned} \quad (5.20)$$

The q_{μ} and y_{ν} in the above are defined in equations (4.22) and (5.17), respectively. We also define new generators H_{μ}

$$D_{\mu} = \delta_{\mu}(1 + H_{\mu}). \quad (5.21)$$

With these new definitions, the algebra relations and the action of the generators on the vacuum are

$$\begin{aligned}
 q_2 H_0 H_1 - H_1 H_0 &= u_2 + v_{10} H_1 + v_{01} H_0 \\
 q_0 H_1 H_2 - H_2 H_1 &= u_0 + v_{21} H_2 + v_{12} H_1 \\
 q_1 H_2 H_0 - H_0 H_2 &= u_1 + v_{02} H_0 + v_{20} H_2
 \end{aligned}
 \tag{5.22}$$

with

$$\begin{aligned}
 \langle 0|H_0|0\rangle &= \langle 0|H_1|0\rangle = \langle 0|H_2|0\rangle = 0 \\
 (\mathcal{L}_0/\delta_0)\langle 0|H_1 &= (\mathcal{L}_1/\delta_1)\langle 0|H_0 & (\mathcal{L}_0/\delta_0)\langle 0|H_2 &= (\mathcal{L}_2/\delta_2)\langle 0|H_0 \\
 (\mathcal{R}_0/\delta_0)H_1|0\rangle &= (\mathcal{R}_1/\delta_1)H_0|0\rangle & (\mathcal{R}_0/\delta_0)H_2|0\rangle &= (\mathcal{R}_2/\delta_2)H_0|0\rangle.
 \end{aligned}
 \tag{5.23}$$

The algebra (5.22) is of a special case of those defined in equation (1.24) [13, 14]. For the latter, one can show that if the coefficients are generic, then for $N \geq 3$ the algebra is finite dimensional. For example, there are 28 independent monomials for $N = 3$. Therefore, one can already understand that the $N = 3$ problem is different from the one for $N = 2$. The next point to note is the role of boundary rates for $N = 3$. They appear explicitly in equation (5.24) but also through the y_μ in the coefficients of the algebra (5.22) (see equation (5.20)).

In the next section we ask the question if the algebra (5.22) has representations in the generic case (when the principal cofactors of the boundary matrices are non-zero) aside from the one-dimensional one, which corresponds to $u_\mu = 0$ and $H_\mu = 0$. We shall show that one can have representations of dimension at most two. In general, it is not possible to have higher dimensional representations because the number of constraints coming from the algebra relations is larger than the number of boundary and bulk rates. In the subsequent sections, we will make a systematic investigation of the cases where some or all of the principal minors vanish.

6. Generic three-state exclusion model: boundary intensity matrices have non-zero minors

In this section we consider the question when the algebra (5.22) has Fock representations defined by equations (5.23)–(5.24) under the assumption that the \mathcal{R}_μ and \mathcal{L}_μ are non-zero. This is a very technical section and the reader not interested in the method of answering the question can proceed directly to the end of the section where the result is given. For our specific purpose it is convenient to rewrite the algebra in a slightly different way using new notation.

Let

$$q_v = Q_v^2 \quad \lambda_i = \frac{\mathcal{L}_i \delta_0}{\mathcal{L}_0 \delta_i} \quad \mu_i = \frac{\mathcal{R}_i \delta_0}{\mathcal{R}_0 \delta_i} \quad \alpha_i^2 = \frac{\lambda_i}{\mu_i} \quad (v = 0, 1, 2; i = 1, 2)
 \tag{6.1}$$

and instead of the generators H_v use L_v defined as follows:

$$D_0 = \delta_0(1 + L_0) \quad D_i = \delta_i(1 + \lambda_i L_1) \quad D_2 = \delta_2(1 + \mu_2 L_2).
 \tag{6.2}$$

The algebra relations are

$$\begin{aligned}
 Q_2^2 L_0 L_1 - L_1 L_0 &= m_2 + n_{10} L_1 + n_{01} L_0 \\
 Q_0^2 L_1 L_2 - L_2 L_1 &= m_0 + n_{21} L_2 + n_{12} L_1 \\
 Q_1^2 L_2 L_0 - L_0 L_2 &= m_1 + n_{02} L_0 + n_{20} L_2
 \end{aligned}
 \tag{6.3}$$

and the conditions for a Fock representation are

$$\begin{aligned} \langle 0|L_1 &= \langle 0|L_0 & \langle 0|L_2 &= \alpha_1^2 \langle 0|L_0 \\ L_2|0\rangle &= L_0|0\rangle & L_1|0\rangle &= \alpha_2^2 L_0|0\rangle \\ \langle 0|L_\nu|0\rangle &= 0 & (\nu &= 0, 1, 2). \end{aligned} \quad (6.4)$$

This algebra depends on 14 parameters, and after setting the time scale we are effectively left with 13. For the physical problem, the m_ν and $n_{\mu\nu}$ are related to the rates:

$$\begin{aligned} m_0 &= \frac{1}{\lambda_1 \mu_2} \left(\frac{y_1 - y_2}{g_{12}} + 1 - Q_0^2 \right) \\ m_1 &= \frac{1}{\mu_2} \left(\frac{y_2 - y_0}{g_{02}} + 1 - Q_1^2 \right) \\ m_2 &= \frac{1}{\lambda_1} \left(\frac{y_0 - y_1}{g_{10}} + 1 - Q_2^2 \right) \end{aligned} \quad (6.5)$$

and

$$\begin{aligned} n_{10} &= 1 - Q_2^2 + \frac{y_0}{g_{10}} & n_{01} &= \frac{1}{\lambda_1} \left(1 - Q_2^2 - \frac{y_1}{g_{10}} \right) \\ n_{02} &= \frac{1}{\mu_2} \left(1 - Q_1^2 + \frac{y_2}{g_{02}} \right) & n_{20} &= \left(1 - Q_1^2 - \frac{y_0}{g_{02}} \right) \\ n_{21} &= \frac{1}{\lambda_1} \left(1 - Q_0^2 + \frac{y_1}{g_{21}} \right) & n_{12} &= \frac{1}{\mu_2} \left(1 - Q_0^2 - \frac{y_2}{g_{21}} \right). \end{aligned} \quad (6.6)$$

At this point two strategies are possible. One can look for vacuum expectation values of words of different lengths and find consistency conditions for the 14 parameters; or one can look for matrix representations of the algebra and in this way obtain consistency conditions among the parameters. We shall do both—we shall take matrix representations and indicate which of the consistency conditions on the matrix elements also come from specific vacuum expectation values of words of a given length. In order to do so, it is convenient to work with parameters $p_{\mu\nu}$ and r_μ introduced below instead of the m_μ and $n_{\mu\nu}$ introduced earlier:

$$\begin{aligned} m_0 &= Q_0 \alpha_1 \alpha_2 N(Q_0 / (\alpha_1 \alpha_2)) r_0 \\ m_1 &= Q_1 \alpha_1 N(Q_1 \alpha_1) r_1 \\ m_2 &= Q_2 \alpha_2 N(Q_2 \alpha_2) r_2 \\ n_{01} &= Q_2 \alpha_2 N(Q_2^2 \alpha_2) p_{01} \\ n_{10} &= Q_2 N(Q_2^2 \alpha_2) p_{10} \\ n_{20} &= Q_1 N(Q_1^2 \alpha_1) p_{20} \\ n_{02} &= Q_1 \alpha_1 N(Q_1^2 \alpha_1) p_{02} \\ n_{21} &= Q_0 \alpha_2 N(Q_0^2 / (\alpha_1 \alpha_2)) p_{21} \\ n_{12} &= Q_0 \alpha_1 N(Q_0^2 / (\alpha_1 \alpha_2)) p_{12}. \end{aligned} \quad (6.7)$$

The constraints on the parameters we shall obtain in what follows look much simpler in terms of the new parameters. In equations (6.7) and in what follows we use the notation

$$M(x) = x + \frac{1}{x} \quad \text{and} \quad N(x) = x - \frac{1}{x}. \quad (6.8)$$

We consider a three-dimensional representation of the algebra equation (6.3). We make this choice since we are going to prove that, in the generic case, the two-dimensional representation is the largest we can have. Using the boundary conditions (6.4) we can bring

the matrix representation of the generators L_μ to a tridiagonal form by means of a similarity transformation by an orthogonal matrix, just as was done for the $N = 2$ case in [9]. One can show that with

$$|0\rangle = (1, 0, 0) \quad \text{and} \quad |0\rangle = (1, 0, 0)^T$$

(where T denotes transposition) the generators then take on the expressions

$$\begin{aligned} L_0 &= \begin{pmatrix} 0 & \sqrt{f_1} & 0 \\ \sqrt{f_1} & c_1^{(0)} & \sqrt{f_2} \\ 0 & \sqrt{f_2} & c_2^{(0)} \end{pmatrix} \\ L_1 &= \begin{pmatrix} 0 & \sqrt{f_1} & 0 \\ \alpha_2^2 \sqrt{f_1} & \alpha_1 c_1^{(1)} & \sqrt{f_2}/Q_2^2 \\ 0 & \alpha_2^2 Q_2^2 \sqrt{f_2} & \alpha_2 c_2^{(1)} \end{pmatrix} \\ L_2 &= \begin{pmatrix} 0 & \alpha_1^2 \sqrt{f_1} & 0 \\ \sqrt{f_1} & \alpha_1 c_1^{(2)} & \alpha_1^2 Q_1^2 \sqrt{f_2} \\ 0 & \sqrt{f_2}/Q_1^2 & \alpha_1 c_2^{(2)} \end{pmatrix}. \end{aligned} \tag{6.9}$$

Notice that a one-dimensional representation is obtained if $f_1 = 0$, a two-dimensional one if $f_2 = 0, f_1 \neq 0$ and a three-dimensional one if $f_1 \neq 0, f_2 \neq 0$. f_i and $c_i^{(j)}$ are still to be determined from the algebra relations, the Fock conditions (6.4) having already been taken into account.

We now insert the 3×3 matrices (6.9) in the algebraic relations (6.3) and from the equality of each matrix element (i, j) on the left- and right-hand sides, get the following equations. From the (1, 1) matrix element, we obtain

$$r_0 = r_1 = r_2 = f_1. \tag{6.10}$$

From the matrix elements (1, 2), (2, 1) and (2, 2) we get

$$\begin{aligned} c_1^{(0)} &= M(\alpha_2 Q_2) p_{10} + M(Q_2) p_{01} = M(Q_1) p_{02} + M(\alpha_1 Q_1) p_{20} \\ c_1^{(1)} &= M(Q_2) p_{10} + M(\alpha_2 Q_2) p_{01} = M(Q_0/(\alpha_1 \alpha_2)) p_{21} + M(Q_0) p_{12} \\ c_1^{(2)} &= M(\alpha_1 Q_1) p_{02} + M(Q_1) p_{20} = M(Q_0/(\alpha_1 \alpha_2)) p_{12} + M(Q_0) p_{21} \tag{6.11} \\ \frac{N(Q_2^3 \alpha_2)}{N(\alpha_2) M(Q_2)} f_2 &= f_1 + p_{10}^2 + p_{01}^2 + M(Q_2^2 \alpha_2) p_{01} p_{10} \\ \frac{N((\alpha_1 \alpha_2)/Q_0^3)}{N(\alpha_1 \alpha_2) M(Q_0)} f_2 &= f_1 + p_{12}^2 + p_{21}^2 + M((\alpha_1 \alpha_2)/Q_0^2) p_{21} p_{12} \\ \frac{N(Q_1^3 \alpha_1)}{N(\alpha_1) M(Q_1)} f_2 &= f_1 + p_{20}^2 + p_{02}^2 + M(Q_1^2 \alpha_1) p_{02} p_{20}. \end{aligned} \tag{6.12}$$

Let us pause here for a moment. Equations (6.10) and (6.11) can also be obtained from the vacuum expectation values of words of length two and three. The one-dimensional representation is obtained from the condition $f_1 = 0$ and the two-dimensional representation is obtained by setting $f_2 = 0$ in (6.12). From equations (6.11) and (6.12) the $p_{\mu\nu}$ are completely determined as are the r_μ (equation (6.10)). This means that in the algebra (6.3), the free parameters are α_1, α_2 and $Q_\mu, \mu = 0, 1, 2$. The next logical step would be to solve the equations and look for solutions with positive rates. This is a difficult exercise. For a given physical problem, where some conditions on the rates are given, the problem is simpler since there are fewer parameters.

We now go on with the rest of the matrix elements (1, 3), (2, 3) and (3, 1)–(3, 3). We get

$$Q_0^2 Q_1^2 Q_2^2 = 1 \quad (6.13)$$

$$\begin{aligned} c_2^{(0)} &= \frac{M(Q_1)}{N(\alpha_1 Q_1^4)} ((N(\alpha_1^2 Q_1^4) - N(Q_1^2)) p_{20} + M(Q_1) N(Q_1 \alpha_1) p_{02}) \\ &= \frac{M(Q_2)}{N(\alpha_2 Q_2^4)} ((N(\alpha_2^2 Q_2^4) - N(Q_2^2)) p_{10} + M(Q_2) N(Q_2 \alpha_2) p_{01}) \\ c_2^{(1)} &= \frac{M(Q_2)}{N(\alpha_2 Q_2^4)} ((N(\alpha_2^2 Q_2^4) - N(Q_2^2)) p_{01} + M(Q_2) N(Q_2 \alpha_2) p_{10}) \\ &= \frac{M(Q_0)}{N(Q_0^4/(\alpha_1 \alpha_2))} \left(\left(N \left(\frac{Q_0^4}{\alpha_1 \alpha_2} \right) - N(Q_0^2) \right) p_{21} + M(Q_0) N \left(\frac{Q_0}{\alpha_1 \alpha_2} \right) p_{12} \right) \\ c_2^{(2)} &= \frac{M(Q_1)}{N(\alpha_1 Q_1^4)} ((N(\alpha_1^2 Q_1^4) - N(Q_1^2)) p_{02} + M(Q_1) N(Q_2 \alpha_2) p_{20}) \\ &= \frac{M(Q_0)}{N(Q_0^4/(\alpha_1 \alpha_2))} \left(\left(N \left(\frac{Q_0^4}{\alpha_1 \alpha_2} \right) - N(Q_0^2) \right) p_{12} + M(Q_0) N \left(\frac{Q_0}{\alpha_1 \alpha_2} \right) p_{21} \right) \end{aligned} \quad (6.14)$$

and

$$\begin{aligned} f_2 &= -f_1 + \frac{1}{N(\alpha_2 Q_2)} (N(Q_2) c_0^{(2)} c_1^{(2)} - N(\alpha_2 Q_2^2) (p_{01} c_0^{(2)} + p_{10} c_1^{(2)})) \\ &= -f_1 + \frac{1}{N(Q_0/\alpha_1 \alpha_2)} (N(Q_0) c_1^{(2)} c_2^{(2)} - N(Q_0^2/(\alpha_1 \alpha_2)) (p_{12} c_1^{(2)} + p_{21} c_2^{(2)})) \\ &= -f_1 + \frac{1}{N(\alpha_1 Q_1)} (N(Q_1) c_0^{(2)} c_2^{(2)} - N(\alpha_1 Q_1^2) (p_{20} c_2^{(2)} + p_{02} c_0^{(2)})). \end{aligned} \quad (6.15)$$

Equations (6.11) and (6.14) give a system of six homogeneous equations for the $p_{\mu\nu}$ with determinant generically different from zero. In order to have a non-trivial solution for the $p_{\mu\nu}$, we must set this determinant to zero, thus introducing an extra condition on the rates. This leaves one of them (say p_{10}) free. The same equations determine $c_i^{(\mu)}$ ($i = 1, 2, \mu = 0, 1, 2$). We are thus left with eight equations: three from (6.12), one from (6.13), three from (6.15), and one determinantal condition for the seven unknowns f_2, p_{10}, Q_μ and α_i . (We can take $f_i = 1$ in equation (6.10) to set the time scale.) This implies that we cannot, generically have a three-dimensional representation. This also implies that the algebra (6.3) with the Fock conditions (6.4) can have representations of dimensions one and two only. The conditions on the parameters are not neat and we found no way to simplify them.

We come now back to the physical problem. The algebra (5.22) contains nine parameters— q_0, q_1, q_2 and the ratios $(y_0/g_{10}), (y_0/g_{02}), (y_1/g_{10}), (y_1/g_{21}), (y_2/g_{02}), (y_2/g_{21})$; the boundary conditions (5.24) depend on four others, the λ_i and μ_i for $I = 1, 2$. This makes a total of 13 parameters. Therefore, for the physical problem, the case of two-dimensional representations stays interesting since the algebraic varieties on which the solution set exists is not trivial.

7. The two non-zero cofactors case

In the last section we considered the generic case where all the cofactors \mathcal{L}_μ and \mathcal{R}_μ are non-zero and we found that the algebra (5.22) had Fock representations defined by equations (5.23) and (5.24) of dimension one and two only. We should have continued our

study and looked at all cases when some or all of the cofactors vanish. In this section we shall confine ourselves only to the case

$$\mathcal{L}_0 = \mathcal{L}_2 = \mathcal{R}_0 = \mathcal{R}_1 = 0 \quad \mathcal{L}_1 \neq 0, \mathcal{R}_2 \neq 0. \tag{7.1}$$

We had in mind the physical application of [5]. The interested reader can easily duplicate the calculations presented in this section for the other cases. With (7.1), the Fock representations have to satisfy the following simple conditions:

$$\langle 0|H_0 = \langle 0|H_2 = 0 \quad H_0|0\rangle = H_1|0\rangle = 0 \quad \langle 0|H_\mu|0\rangle = 0. \tag{7.2}$$

In order to find out in which cases the algebra (5.22) has Fock representations defined by equation (7.2), we take vacuum expectation values for monomials of different degree.

Monomials of degree two (consider the algebra relations between $\langle 0|$ and $|0\rangle$) give

$$u_1 = u_2 = 0. \tag{7.3}$$

Monomials of degree three (consider $\langle 0|H_1H_0H_2|0\rangle$ and commute H_0 through to the left and the right) give

$$v_{10} = v_{20} = v. \tag{7.4}$$

Monomials of degree four $\langle 0|H_1H_1H_0H_2|0\rangle$ and $\langle 0|H_1H_2H_0H_2|0\rangle$ give

$$q_0v_{01} = q_2v_{21} \quad \text{and} \quad q_0v_{02} = q_1v_{12} \tag{7.5}$$

respectively. We have to distinguish several cases

(a) v_{12} and v_{21} are non-zero. Monomials of degree five (consider $\langle 0|H_1H_0H_1H_2H_2|0\rangle$) give

$$q_1 = q_2 = q. \tag{7.6}$$

Taking into account relations (7.3)–(7.6) the algebra (5.22) becomes

$$\begin{aligned} qH_0H_1 - H_1H_0 &= vH_1 + \frac{q}{q_0}v_{21}H_0 \\ qH_2H_0 - H_0H_2 &= \frac{q}{q_0}v_{12}H_0 + vH_2 \\ q_0H_1H_2 - H_2H_1 &= u_0 + v_{12}H_1 + v_{21}H_2. \end{aligned} \tag{7.7}$$

It is useful to change variables

$$H_0 = -v\mathcal{N} \quad H_1 = \frac{u_0}{v_{12}}\mathcal{A} \quad H_2 = \frac{u_0}{v_{21}}\mathcal{B} \quad z_2 = \frac{u_0q_0}{v_{12}v_{21}} \tag{7.8}$$

in terms of which the algebra becomes

$$z_2 \left(\mathcal{A}\mathcal{B} - \frac{1}{q_0}\mathcal{B}\mathcal{A} \right) = 1 + \mathcal{A} + \mathcal{B} \tag{7.9a}$$

$$\mathcal{A}\mathcal{N} - q\mathcal{N}\mathcal{A} = \mathcal{A} - \frac{q}{z_2}\mathcal{N} \tag{7.9b}$$

$$\mathcal{N}\mathcal{B} - q\mathcal{B}\mathcal{N} = \mathcal{B} - \frac{q}{z_2}\mathcal{N}. \tag{7.9c}$$

This algebra has the representations

$$\langle 0|\mathcal{N} = 0 = \mathcal{N}|0\rangle \quad \langle 0|\mathcal{B} = 0 = \mathcal{A}|0\rangle. \tag{7.10}$$

We can now use the results of [9] to find the following representation of (7.9a):

$$\mathcal{A} = \begin{pmatrix} a_1 & f_1 & 0 & 0 & \dots \\ 0 & a_2 & f_2 & 0 & \dots \\ 0 & 0 & a_3 & f_3 & \dots \\ 0 & 0 & 0 & a_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \mathcal{B} = \mathcal{A}^T = \begin{pmatrix} a_1 & 0 & 0 & 0 & \dots \\ f_1 & a_2 & 0 & 0 & \dots \\ 0 & f_2 & a_3 & 0 & \dots \\ 0 & 0 & f_3 & a_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (7.11)$$

where

$$a_n = z_2^{-1} \{n-1\}_\lambda \quad f_n^2 = z_2^{-2} \{n\}_\lambda (z_2 + \{n-1\}_\lambda) \\ \{n\}_\lambda = \frac{1-\lambda^n}{1-\lambda} \quad n \geq 1 \quad \{0\}_\lambda = 1 \quad \lambda = q_0^{-1}. \quad (7.12)$$

We have introduced the symbol z_2 in keeping with the notation of [9]. \mathcal{A}^T is the transpose of matrix \mathcal{A} . There are two cases when we can have representations for \mathcal{N} in equations (7.9):

(i)

$$q = q_0 \quad v_{01} = v_{21} \quad v_{02} = v_{12} \quad (7.13)$$

when

$$\mathcal{N} = \begin{pmatrix} h_1 & 0 & 0 & \dots \\ 0 & h_2 & 0 & \dots \\ 0 & 0 & h_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (7.14)$$

with

$$h_n = \{n-1\}_q \quad (7.15)$$

(ii)

$$q = 0, q_0 \neq 0, v_{01} = v_{02} = 0 \quad (7.16)$$

with

$$h_n = \{n-1\}_{q \rightarrow 0} = 1 - \delta_{n1}. \quad (7.17)$$

(b) We consider the case (see equation (7.5))

$$v_{01} = v_{02} = v_{12} = v_{21} = 0 \quad v_{10} = v_{20} = v \quad u_1 = u_2 = 0. \quad (7.18)$$

Monomials of degree five give (see equation (7.6))

$$q_1 = q_2 = q. \quad (7.19)$$

With (7.18) and (7.19) the algebra (5.22) becomes

$$qH_0H_1 - H_1H_0 = vH_1 \\ qH_2H_0 - H_0H_2 = vH_2 \\ q_0H_1H_2 - H_2H_1 = u_0. \quad (7.20)$$

We make the transformation

$$H_0 = -v\mathcal{N} \quad H_1 = \sqrt{\frac{u_0}{q_0}}\mathcal{A} \quad H_2 = \sqrt{\frac{u_0}{q_0}}\mathcal{B} \quad \lambda = \frac{1}{q_0} \quad (7.21)$$

and obtain the algebra relations

$$\mathcal{A}\mathcal{B} - \lambda\mathcal{B}\mathcal{A} = 1 \quad \mathcal{A}\mathcal{N} - q\mathcal{N}\mathcal{A} = \mathcal{A} \quad \mathcal{N}\mathcal{B} - q\mathcal{B}\mathcal{N} = \mathcal{B} \quad (7.22)$$

and

$$\mathcal{A}|0\rangle = \mathcal{N}|0\rangle = 0 = \langle 0|\mathcal{B} = \langle 0|\mathcal{N}. \tag{7.23}$$

The Fock representation of this algebra (matrices satisfying (7.22) and (7.23)) is

$$\mathcal{A} = \mathcal{B}^T = \begin{pmatrix} 0 & g_1 & 0 & \dots \\ 0 & 0 & g_2 & \\ 0 & 0 & 0 & \\ \vdots & & & \ddots \end{pmatrix} \quad \mathcal{N} = \begin{pmatrix} h_1 & 0 & 0 & \dots \\ 0 & h_2 & 0 & \\ 0 & 0 & h_3 & \\ \vdots & & & \ddots \end{pmatrix} \tag{7.24}$$

where

$$g_n^2 = \{n\}_\lambda \quad h_n = \{n-1\}_q. \tag{7.25}$$

(c) We consider the case (see equation (7.5))

$$v_{21} = v_{01} = 0.$$

The algebra (5.22) again with $q_1 = q_2 = q$ becomes

$$\begin{aligned} qH_0H_1 - H_1H_0 &= vH_1 \\ qH_2H_0 - H_0H_2 &= \frac{q}{q_0}v_{12}H_0 + vH_2 \\ q_0H_1H_2 - H_2H_1 &= u_0 + v_{12}H_1. \end{aligned} \tag{7.26}$$

We make the substitutions

$$H_0 = -v\mathcal{N} \quad H_1 = \frac{u_0}{v_{12}}\mathcal{A} \quad H_2 = \frac{v_{12}}{q_0}\mathcal{B} \quad \lambda = \frac{1}{q_0} \tag{7.27}$$

to obtain the algebra

$$\mathcal{A}\mathcal{B} - \lambda\mathcal{B}\mathcal{A} = 1 + \mathcal{A} \quad \mathcal{A}\mathcal{N} - q\mathcal{N}\mathcal{A} = \mathcal{A} \quad \mathcal{N}\mathcal{B} - q\mathcal{B}\mathcal{N} = \mathcal{B} - q\mathcal{N}. \tag{7.28}$$

This algebra has the representations

$$\mathcal{A} = \begin{pmatrix} 0 & f_1 & 0 & \dots \\ 0 & 0 & f_2 & \\ 0 & 0 & 0 & \\ \vdots & & & \ddots \end{pmatrix} \quad \mathcal{B} = \begin{pmatrix} b_1 & 0 & 0 & \dots \\ f_1 & b_2 & 0 & \\ 0 & f_2 & b_3 & \\ \vdots & & & \ddots \end{pmatrix} \tag{7.29}$$

where

$$f_n^2 = \{n\}_\lambda \quad b_n = \{n-1\}_\lambda \tag{7.30}$$

and if:

(i)

$$q = q_0 = \lambda^{-1}$$

then

$$\mathcal{N} = \begin{pmatrix} h_1 & 0 & 0 & \dots \\ 0 & h_2 & 0 & \\ 0 & 0 & h_3 & \\ \vdots & & & \ddots \end{pmatrix} \tag{7.31}$$

where

$$h_k = \{k-1\}_\lambda \tag{7.32}$$

(ii)

$$q = 0$$

which implies $v_{02} = 0$ one has to take

$$h_k = \{k - 1\}_{\lambda \rightarrow 0} = 1 - \delta_{k1}. \tag{7.33}$$

In order to help the reader, we shall summarize the results. The algebra (5.22) has Fock representations defined by

$$\langle 0|H_2 = \langle 0|H_0 = 0 = H_0|0\rangle = H_1|0\rangle \tag{7.34}$$

if (necessary condition)

$$u_1 = u_2 = 0 \quad v_{10} = v_{20} = v \quad q_1 = q_2 = q \tag{7.35}$$

in which case the algebra relations become

$$\begin{aligned} qH_0H_1 - H_1H_0 &= vH_1 + v_{01}H_0 \\ qH_2H_0 - H_0H_2 &= v_{02}H_0 + vH_2 \\ q_0H_1H_2 - H_2H_1 &= u_0 + v_{12}H_1 + v_{21}H_2. \end{aligned} \tag{7.36}$$

This algebra has Fock representations in five cases

(1)

$$v_{01} = v_{21} \quad v_{02} = v_{12} \quad q_0 = q. \tag{7.37}$$

With the notation

$$H_0 = -v\mathcal{N} \quad H_1 = \frac{u_0}{v_{12}}\mathcal{A} \quad H_2 = \frac{u_0}{v_{21}}\mathcal{B} \quad z_2 = \frac{u_0q_0}{v_{12}v_{21}}$$

\mathcal{A} and \mathcal{B} are given by equation (7.11)–(7.12) with $\lambda = q_0^{-1}$ and \mathcal{N} by equations (7.14) with (7.15).

(2)

$$v_{01} = v_{02} = q = 0. \tag{7.38}$$

Same as case (1) except that \mathcal{N} is given by equation (7.14) with (7.17).

(3)

$$v_{01} = v_{02} = v_{12} = v_{21} = 0. \tag{7.39}$$

$$H_0 = -v\mathcal{N} \quad H_1 = \sqrt{\frac{u_0}{q_0}}\mathcal{A} \quad H_2 = \sqrt{\frac{u_0}{q_0}}\mathcal{B} \quad \lambda = \frac{1}{q_0}. \tag{7.40}$$

\mathcal{A} , \mathcal{B} and \mathcal{N} are given by equation (7.24) and (7.25).

(4)

$$\begin{aligned} v_{01} = v_{21} = 0 \quad q = q_0 = \frac{1}{\lambda} \\ H_0 = -v\mathcal{N} \quad H_1 = \frac{u_0}{v_{12}}\mathcal{A} \quad H_2 = \frac{v_{12}}{q_0}\mathcal{B} \quad \lambda = \frac{1}{q_0}. \end{aligned} \tag{7.41}$$

\mathcal{A} , \mathcal{B} and \mathcal{N} are given by equation (7.29) and \mathcal{N} by equation (7.31) with (7.32).

(5)

$$v_{02} = v_{01} = v_{21} = 0 \quad q_0.$$

Everything as in case (4) except \mathcal{N} is given by equation (7.31) with equation (7.33).

This closes the problem of Fock representation of the algebra (7.7). We now turn to the physical problem. This implies using the definitions (5.19) and (5.20) of the parameters u_μ and $u_{\mu\nu}$. We first have to give the boundary rates which give equation (7.1). Using (2.7) we have the solution

$$\begin{aligned} L_2^1 &= L_0^1 = R_1^2 = R_0^2 = 0 \\ \mathcal{L}_1 &= L_1^0 L_0^2 + L_1^2 L_1^0 + L_1^2 L_2^0 \\ \mathcal{R}_2 &= R_0^1 R_2^0 + R_2^1 R_1^0 + R_2^1 R_2^0. \end{aligned} \tag{7.42}$$

We now compute \mathcal{B}_μ and x_μ using equations (5.8) and (5.9):

$$\begin{aligned} \delta_0 &= \xi(R_0^1 L_1^2 + L_0^2 R_0^1 + L_0^2 R_2^1) \\ \delta_1 &= \xi(R_1^0 L_0^2 + R_1^0 L_1^2 + R_2^0 L_1^2 + \mathcal{L}_1) \\ \delta_2 &= \xi(R_0^1 L_2^0 + R_2^1 L_2^0 + R_2^1 L_1^0 + \mathcal{R}_2) \end{aligned} \tag{7.43}$$

and

$$\begin{aligned} x_0 &= \xi(L_0^2 \mathcal{R}_2 - R_0^1 \mathcal{L}_1) \\ x_1 &= \xi(L_1^2 \mathcal{R}_2 + (R_0^1 + R_2^1) \mathcal{L}_1) \\ x_2 &= -\xi((L_0^2 + L_1^2) \mathcal{R}_2 + R_2^1 \mathcal{L}_1). \end{aligned} \tag{7.44}$$

We now implement the conditions (7.35) to get

$$y_0 = 0 \quad y_1 = g_{10}(1 - q) \quad y_2 = -g_{02}(1 - q). \tag{7.45}$$

We have to consider the five cases separately.

(1) From (7.37) and (5.20) we get

$$y_1 = 0 = y_2. \tag{7.46}$$

Taking into account the positivity of the rates, the definition (5.17) and the relation (5.6) this can be excluded.

(2) From equation (7.38) and (5.19) we have

$$g_{20} = 0 = g_{01} \quad y_0 = 0 \quad y_1 = g_{10} \quad y_2 = -g_{02}. \tag{7.47}$$

Using the definitions (5.19) and (5.20) we get

$$v_{10} = v_{20} = 1$$

and

$$v_{12} = 1 - q_0 + \frac{g_{02}}{g_{21}} \quad v_{21} = 1 - q_0 + \frac{g_{10}}{g_{21}} \quad u_0 = 1 - q_0 + \frac{g_{10} + g_{20}}{g_{21}}$$

and the following constraints on the rates coming from the definition (5.17) of y_μ and from equation (7.47):

$$\begin{aligned} L_0^2 \mathcal{R}_2 &= R_0^1 \mathcal{L}_1 \\ g_{10} &= \frac{L_1^2 \mathcal{R}_2 + (R_0^1 + R_2^1) \mathcal{L}_1}{R_1^0 L_0^2 + R_1^0 L_1^2 + R_2^0 L_1^2 + \mathcal{L}_1} \\ g_{02} &= \frac{(L_0^2 + L_1^2) \mathcal{R}_2 + R_2^1 \mathcal{L}_1}{R_0^1 L_2^0 + R_2^1 L_2^0 + R_2^1 L_1^0 + \mathcal{R}_2}. \end{aligned} \tag{7.48}$$

(3) As a consequence of equations (7.39) and (7.45) we find

$$g_{10} = g_{02} \quad g_{01} = g_{20} \quad g_{12} - g_{21} = g_{10} - g_{01} \tag{7.49}$$

$$y_0 = 0 \quad y_1 = -y_2 = g_{10} - g_{01} \tag{7.50}$$

and

$$v_{12} = 0 = v_{21} \quad u_0 = q_0 - 1 \quad v_{10} = v_{20} = 1 - q. \tag{7.51}$$

From equation (7.50) we find the following constraints on the rates:

$$L_0^2 \mathcal{R}_2 = R_0^1 \mathcal{L}_1$$

$$\frac{L_1^2 \mathcal{R}_2 + (R_0^1 + R_2^1) \mathcal{L}_1}{R_1^0 L_0^2 + R_1^0 L_1^2 + R_2^0 L_1^2 + \mathcal{L}_1} = \frac{(L_0^2 + L_1^2) \mathcal{R}_2 + R_2^1 \mathcal{L}_1}{R_0^1 L_2^0 + R_2^1 L_2^0 + R_2^1 L_1^0 + \mathcal{R}_2} = g_{10} - g_{01}. \tag{7.52}$$

(4) No solutions if we want to maintain positivity of the rates.

(5) We find

$$g_{21} + g_{10} = g_{12} \quad g_{20} = 0 = g_{01} \tag{7.53}$$

$$y_0 = 0 \quad y_1 = g_{10} \quad y_2 = -g_{02} \tag{7.54}$$

and

$$v_{10} = v_{20} = 1 \quad v_{12} = \frac{g_{02} - g_{10}}{g_{21}} \quad u_0 = \frac{g_{02}}{g_{21}}. \tag{7.55}$$

The conditions on the rates can be derived from equations (7.54) and (5.17). Notice that this case is a specialization of case (3). The representation is different because z_2 defined in equation (7.8) and used for case (3) diverges ($v_{21} = 0$).

In the appendix we consider the case of a CP -invariant stochastic process [5]. In this problem, the index ‘0’ denotes vacancies, ‘1’ is the index for particles and ‘2’ for anti-particles. This implies supplementary conditions on the rates. If we are interested in the case where the CP -symmetry is explicitly broken [41, 42], one can use the solutions corresponding to the cases (2), (3) and (5). In all three cases the D_α have the form

$$D_\alpha = x_\alpha E_\alpha (x_\alpha \neq 0) \quad D_\alpha = E_\alpha (x_\alpha = 0) \tag{7.56}$$

where E_α , which can be extracted from the formulae given above and are given explicitly in [25], depend only on the bulk rates. The boundary rates enter in the expressions of x_α and through the fact that they have to ‘match’ the bulk rates (see equations (7.48) and (7.52)). This observation allows us to see the important parameters of the problem.

Before closing this section, let us note that the case where only the cofactors \mathcal{L}_0 and \mathcal{R}_0 vanish can be treated in a similar way. In this case, the algebras satisfied by the \mathcal{A} and \mathcal{B} matrices (see equations (7.9a) and (7.28)) contain \mathcal{A}^2 and \mathcal{B}^2 terms as well. Fock representations for this case are also known [9].

8. Lower rank boundary intensity matrices

In sections 6 and 7 the Fock representations were defined by four conditions (see equations (5.24), (6.4) or (7.2)). The number of conditions depends on the rank of the boundary intensity matrices. In principle, one should examine all the possibilities enumerated in equations (4.6)–(4.12) for both left and right intensity matrices. This is a long exercise. The main point is that once either L_μ^v or R_μ^v or both have all the principal minors to be zero, the number of conditions is smaller and one has more freedom for the bulk rates. We shall give only four illustrative examples in which the physics can be easily understood.

(a) $\mathcal{L}_1 \neq 0$, all other cofactors zero. We take the following to be the boundary intensity matrices:

$$L = \begin{pmatrix} -L_1^0 - L_2^0 & 0 & L_0^2 \\ L_1^0 & 0 & L_1^2 \\ L_2^0 & 0 & -L_1^2 - L_0^2 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} -R_1^0 & R_0^1 & 0 \\ R_1^0 & -R_0^1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{8.1}$$

We set $\xi = 1$ in equations (5.8) and (5.9) to get

$$\begin{aligned} x_0 &= -R_0^1 \mathcal{L}_1 & x_1 &= R_0^1 \mathcal{L}_1 & x_2 &= 0 \\ \delta_0 &= R_0^1 (L_1^2 + L_0^2) & \delta_1 &= R_1^0 (L_1^2 + L_0^2) + \mathcal{L}_1 & \delta_2 &= R_0^1 L_2^0. \end{aligned} \tag{8.2}$$

We get several constraints on the various parameters v_{ij} and u_k that appear in the algebra (5.22) and are given in terms of the bulk and boundary rates in (5.19) and (5.20). From the condition $x_2 = 0$, and by sandwiching (5.22) between $\langle 0|$ and $|0\rangle$, we obtain

$$v_{21} = u_0 \quad v_{20} = u_1 = 0.$$

From $\langle 0|H_1H_2H_0H_2|0\rangle$ we obtain

$$v_{10} = 0$$

from which, on calculating $\langle 0|H_1H_0H_2|0\rangle$, we get $q_1 = q_0 = q$. The correlator $\langle 0|H_1H_0H_1H_2|0\rangle$ gives

$$q_2 = 1 + \frac{v_{01}}{u_0}(q - 1)$$

so we end up with

$$\begin{aligned} \frac{x_0}{\delta_0} &= (g_{01} - g_{10}) = (g_{02} - g_{20}) = (g_{21} - g_{12}) \\ v_{12} &= v_{02} = (1 - q). \end{aligned} \tag{8.3}$$

The algebra is of a similar form as case (c) of section 7 (equation (7.26)), apart from a relabelling of the indices and the presence of an additional constant term:

$$\begin{aligned} qH_1H_2 - H_2H_1 &= v_{21}H_2 + (1 - q)H_1 + v_{21} \\ qH_2H_0 - H_0H_2 &= (1 - q)H_0 \\ q_2H_0H_1 - H_1H_0 &= v_{01}H_0 + u_2. \end{aligned} \tag{8.4}$$

Introducing

$$\begin{aligned} H_0 &= \frac{u_2}{v_{01}} \mathcal{A} & H_1 &= v_{21} \mathcal{B} & H_2 &= (q - 1) \mathcal{N} \\ \lambda &= q_2^{-1} & \text{and} & & \gamma &= -\frac{g_{02}}{g_{01}} \end{aligned} \tag{8.5}$$

we get (compare with equation (7.28)):

$$\begin{aligned} \mathcal{A}\mathcal{B} - \lambda\mathcal{B}\mathcal{A} &= \gamma(1 + \mathcal{A}) \\ \mathcal{A}\mathcal{N} - q\mathcal{N}\mathcal{A} &= \mathcal{A} \\ \mathcal{N}\mathcal{B} - q\mathcal{B}\mathcal{N} &= \mathcal{B} - \mathcal{N} + \frac{1}{1 - q}. \end{aligned} \tag{8.6}$$

Also, instead of four Fock conditions as in section 7, we have only three:

$$\langle 0|\mathcal{A} = \langle 0|\mathcal{N} = 0 \quad \mathcal{B}|0\rangle = \alpha\mathcal{A}|0\rangle \tag{8.7}$$

with

$$\alpha = \left(\frac{g_{02}\delta_0}{x_0(\delta_1 - \delta_0)} \right) \left(\frac{\delta_0 R_1^0}{\delta_1 R_1^1} \right). \tag{8.8}$$

We have not looked for explicit representations of this algebra.

(b) *All the cofactors are zero.* We consider the case when only the following four boundary rates do not vanish:

$$L_2^0 \quad L_0^2 \quad R_0^1 \quad \text{and} \quad R_1^0. \tag{8.9}$$

We find from (5.9)

$$x_\mu = 0 \quad (\mu = 0, 1, 2) \quad (8.10)$$

and the algebra is a known quadratic algebra (see section 3)

$$q_0 D_1 D_2 = D_2 D_1 \quad q_1 D_2 D_0 = D_0 D_2 \quad q_2 D_0 D_1 = D_1 D_0 \quad (8.11)$$

with only two conditions:

$$D_1|0\rangle = \frac{R_1^0}{R_0^1} D_0|0\rangle \quad \text{and} \quad \langle 0|D_2 = \frac{L_2^0}{L_0^2} \langle 0|D_0. \quad (8.12)$$

All vacuum expectation values of degree L can be expressed in terms of $\langle 0|D_0^L|0\rangle$. The appearance of the polynomial algebras in this new context is very interesting since, as shown in section 3, we know plenty of them, and not just for the simple exclusion processes. This opens up the possibility of a new class of solutions for more general processes with open ends.

(c) *All the cofactors are zero, but the non-zero boundary rates are*

$$L_1^0 \quad L_2^0 \quad R_0^1 \quad \text{and} \quad R_1^0. \quad (8.13)$$

In this case the algebra is also given by (8.11) and $x_\mu = 0$, ($\mu = 0, 1, 2$), but the generators satisfy the conditions

$$\langle 0|D_0 = 0 \quad \text{and} \quad D_1|0\rangle = \frac{R_1^0}{R_0^1} D_0|0\rangle. \quad (8.14)$$

It is interesting to observe that no boundary condition appears for the operator D_2 . Using (8.11) and (8.14) it is simple to see that the ground state will have the single word D_2^L , which corresponds to the lattice filled by particle ‘2’. This can be expected physically from (8.13) since the species ‘2’ is never removed from the chain at either end, unlike the species ‘0’ and ‘1’, and is created at the left end with a certain rate $L_2^0 \neq 0$. Therefore, when the time goes to infinity the system will end up with only particles of species ‘2’.

(d) *All cofactors are zero, but the non-zero boundary rates are*

$$L_1^0 \quad L_2^0 \quad R_1^0 \quad \text{and} \quad R_2^0. \quad (8.15)$$

The algebra is again by (8.11) but the conditions are

$$\langle 0|D_0 = D_0|0\rangle = 0 \quad (8.16)$$

with no conditions for D_1 and D_2 . In this case, we have L degenerate ground states expressed in terms of the words $D_1^n D_2^{L-n}$ ($n = 1, 2, \dots, L$). This is expected physically from (8.15) since, in the large time limit, we should expect no ‘0’ particles and the particles ‘1’ and ‘2’ are conserved separately.

9. Conclusions

This was a long journey. We will first list the mathematical results which are independent of the physical applications. In section 3 we have given a list of polynomial algebras defined by a set of homogeneous quadratic relations (see, for example (3.25)–(3.27)). In sections 6 and 7 we have given Fock representations for the algebra defined by equation (5.22). These are a special class of the inhomogeneous quadratic algebras defined by equation (1.24). The representations are one-dimensional (see (5.16)), two-dimensional for the ‘generic’ case described in section 6, or infinite-dimensional (see the five special cases of the algebra (7.36)). These algebras can be useful for physical applications other than for the case of open chains considered in this paper. For example, we used them ourselves while

studying stochastic processes on a ring [25]. As for the physical applications relevant to this paper, we have looked for steady states of stochastic processes defined by bulk and boundary intensity matrices (see (1.1)). The probability distribution of the steady states is given by the expression (1.14) for chains with open ends.

In our approach the same expression without $|0\rangle$ is used for a left open end and a closed right end. In the case of closed chains we have to drop the $\langle 0|$ and $|0\rangle$ symbols. Polynomial algebras are defined by taking the X 's equal to zero in equation (1.12). They exist only for special choices of the bulk rates. A given polynomial algebra can be used in three places: for closed ends (see section 3), for the left end open and the right end closed when supplementary restrictions come from the first equation in (1.13) with the X taken to be zero (see section 4) and even for the case of a chain with open ends (see, the example, in section 8, (8.11) and (8.12)). As far as we know, it is the first time that the problem with one or two closed ends has been solved using algebraic methods.

Most of this paper is dedicated to the problem of the chain with open ends where we have taken c -numbers for the X in equations (1.12) and (1.13). We have confined ourselves to the simpler problem of exclusion processes. In solving this problem a crucial point is the rank of the boundary intensity matrices, since they dictate the number of conditions which define the Fock representations. In the three-state problem, they can be four (see sections 6 and 7) three or two (see section 8). In the case of four conditions an important role is played by the number of cofactors of the boundary intensity matrices which vanish. All in all, we have given many examples which can be used for physical applications. It is obvious how the present method may be generalized to four- or more-state problems. The case of periodic boundary conditions is considered in [25].

After writing this paper, one can ask where the mathematical beauty lies: probably in the simpler cases where polynomial algebras can be used and in the algebras described in section 7. The problem of open boundaries with two states depends on five parameters; the one with three states on 17 parameters and there is a price to pay in order to find solutions. In this paper we have never properly exploited the Krebs–Sandow theorem (equations (1.12)–(1.15)) since we have always chosen representations of the X_α to be scalars or zero. The reason for this is that we wanted to have quadratic algebras that are understood, like the polynomial algebras or those of type (1.24), and we did not find new ones.

We would like to make one last comment which has to do with the connection between finding the ground states by algebraic methods and the integrability of the Hamiltonian. As is shown in section 3, several exactly integrable Hamiltonians, connected with stochastic dynamics, can have their ground-state wavefunction expressed in terms of words defined by algebraic relations. However, in general, the existence of an algebraic form of the ground-state wavefunction is not a sufficient condition in order to ensure exact integrability.

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Appendix. The CP -invariant steady state

It was recently shown that in steady states CP -invariance can be spontaneously broken [5] and that the phase diagram cannot be obtained by mean-field methods [25]. This is not the right place to describe the physics of the problem and the interested reader should have a look at the references. The model where this invariance is spontaneously broken is a special case of the problem studied in section 7, where the rates we choose are CP -invariant. Charge conjugation (C) is defined by the operation $1 \rightarrow 2$, $2 \rightarrow 1$ and $0 \rightarrow 0$, while the parity (P) operator corresponds to interchanging left and right.

The non-vanishing boundary rates are

$$\begin{aligned} L_1^2 = R_2^1 & & L_1^0 = R_2^0 & & L_2^0 = R_1^0 & & L_0^2 = R_0^1 \\ \mathcal{L}_1 = \mathcal{R}_2 & & \mathcal{L}_0 = \mathcal{L}_2 = \mathcal{R}_0 = \mathcal{R}_1 = 0. \end{aligned} \quad (\text{A.1})$$

The bulk rates are

$$g_{10} = g_{02} \quad g_{01} = g_{20} \quad q_1 = q_2 = q \quad (\text{A.2})$$

and g_{12} and g_{21} are arbitrary.

In [5] a special choice was taken:

$$L_1^0 = R_2^0 = \alpha \quad \text{and} \quad L_0^2 = R_0^1 = \beta \quad (\text{A.3})$$

with all other boundary rates set to zero. The non-vanishing bulk rates were taken to be $g_{12} = g_{10} = g_{02}$. Here we consider the general case.

From equations (7.43), (7.44) and (5.17) and choosing $\xi^{-1} = (L_0^2 + 2L_1^2)$ we obtain

$$\begin{aligned} \delta_0 = L_0^2 & & \delta_1 = \delta_2 = (L_1^0 + L_2^0) \\ y_0 = 0 & & y_1 = -y_2 = \frac{L_1^0 L_0^2}{L_1^0 + L_2^0} + L_1^2. \end{aligned} \quad (\text{A.4})$$

We now use the results of section 7 (equations (7.43)–(7.55)). From equation (7.45) and (A.4) we have

$$\frac{L_1^0 L_0^2}{L_1^0 + L_2^0} + L_1^2 = g_{10} - g_{01}. \quad (\text{A.5})$$

We now consider the cases (2), (3) and (5) separately.

Case (2), $g_{20} = g_{01} = 0$. We use equations (7.47) and (7.48) to get

$$\frac{L_1^0 L_0^2}{L_1^0 + L_2^0} + L_1^2 = g_{10} \quad (\text{A.6})$$

and

$$\begin{aligned} v_{10} = v_{20} = 1 & & v_{01} = v_{02} = 0 \\ v_{12} = v_{21} = \frac{g_{10} + g_{21} - g_{12}}{g_{21}} & & u_0 = \frac{2g_{10} + g_{21} - g_{12}}{g_{21}}. \end{aligned} \quad (\text{A.7})$$

Using equation (5.21) it is easy to show that

$$D_1 = \left(\frac{L_1^0 + L_2^0}{L_0^2} \right) (1 + H_1) \quad D_2 = \left(\frac{L_1^0 + L_2^0}{L_0^2} \right) (1 + H_2). \quad (\text{A.8})$$

Introducing

$$\lambda = \frac{1}{q_0} = \frac{g_{21}}{g_{12}} \quad \text{and} \quad \omega = \frac{g_{10}}{g_{12}} \quad (\text{A.9})$$

we have

$$H_1 = H_2^T = \sqrt{2\omega + \lambda - 1} \begin{pmatrix} u_1 & v_1 & 0 & \dots \\ 0 & u_2 & v_2 & \dots \\ 0 & 0 & u_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad D_0 = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (\text{A.10})$$

where

$$v_n^2 = \{n\}_\lambda \left(1 + \frac{(\omega + \lambda - 1)^2}{(2\omega - 1 + \lambda)} \{n - 1\}_\lambda \right)$$

$$u_n = \frac{(\omega + \lambda - 1)}{\sqrt{2\omega - 1 + \lambda}} \{n - 1\}_\lambda. \quad (\text{A.11})$$

Notice that the boundary conditions appear only in (A.6) and in the normalization factors of D_1 and D_2 . They do not change the physics of the problem which is governed by ω and λ . In other words, taking only the rates (A.3) or the general case (A.1) does not change the physical results.

In [5] only the case $\lambda = 0$ was considered.

Case (3), $g_{10} = g_{02}$, $g_{01} = g_{20}$ and $g_{12} - g_{21} = g_{10} - g_{01}$. We make use of equations (7.49)–(7.52). The bulk rates and the boundary rates are related by the constraint

$$g_{10} - g_{01} = \frac{L_1^0 L_2^0}{L_1^0 + L_2^0} + L_1^2. \quad (\text{A.12})$$

We then have

$$D_1 = \frac{L_1^0 + L_2^0}{L_2^0} (1 + R) \quad D_2 = D_1^T$$

$$R = \begin{pmatrix} 0 & r_1 & 0 & \dots \\ 0 & 0 & r_2 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad D_0 = \begin{pmatrix} s_1 & 0 & 0 & \dots \\ 0 & s_2 & 0 & \dots \\ 0 & 0 & s_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (\text{A.13})$$

where

$$r_k^2 = 1 - \lambda^k \quad \text{and} \quad s_k = 1 - \{k - 1\}_q \quad (\text{A.14})$$

with

$$\lambda = \frac{g_{21}}{g_{12}} \quad \text{and} \quad q = \frac{g_{20}}{g_{02}} = \frac{g_{01}}{g_{10}}. \quad (\text{A.15})$$

Notice once more that the boundary terms are not essential and in this case as well, the conditions (A.3) capture the whole physics.

Case (5), (see equations (7.53)–(7.55)). Since $g_{02} = g_{10}$, we get $v_{12} = 0$ (see equation (7.55)) which together with (7.41) brings us back to case (3).

In conclusion, the algebraic approach is possible only if certain conditions are satisfied by the bulk rates. The boundary rates have to ‘match’ the bulk rates (see, for example, equations (A.5) and (A.6)). In particular, if the bulk rates are chosen as in [5] the physics will be the same if we have four boundary rates as in this paper or only two as in [5]. What we have shown, however, is that we can consider bulk rates other than those in [5]. For more on this subject see [42].

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